The Standard Series of Mathematics

THE ESSENTIALS OF ALGEBRA

FOR SECONDARY SCHOOLS

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PREFACE.

In the preparation of this book the authors have made an earnest effort to retain all the essentials of the older Algebra text-books, and to introduce and properly emphasize certain newer features which the mathematical studies of the present demand.

The following are some of the special characteristics of the book:

1. **The Number System.** The number system is presented in the first chapter, and from the arithmetical system extension is made to the algebraic number system. In this way the idea of negative number is introduced and the fundamental operations are explained.

2. **Factoring.** This subject is treated with particular fullness, and use is made of the factorial method wherever applicable in the study of Algebra. At the first reading, Sections 79, 80, and 86, covering certain details of factoring, may be omitted if thought desirable. The ordinary student, however, should have no special difficulty in mastering these sections.

3. **The Graph.** The work with graphs is made an integral part of the book. The graphs of simple and quadratic equations are used freely to aid the pupil's understanding of the solutions involved. Graphic illustrations are given wherever it is thought they will make the subject clearer.

4. **Type Forms.** Type forms play an important part in the study of Algebra. The work of the student is greatly simplified if he learns early in his course to recognize and to understand these types. Type forms are extensively used in multiplication, division, factoring, and equations.

5. **Exercises.** The exercises have been selected with a view of clarifying the text and enforcing fundamental principles. They are numerous, and are difficult enough to call for effort on the part of the student.

It is believed that the book contains sufficient matter to furnish a thorough training in the elements of Algebra and to meet the entrance requirements of American colleges.

CONTENTS.

THE

ESSENTIALS OF ALGEBRA.

CHAPTER I.

INTRODUCTION.

1. The Integral Number System is that orderly succession by *ones* which we first learn by counting. We are familiar with it in the Arabic numeral form of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and so on. The characters 1, 2, 3, 4, etc., are symbols of number, but we shall hereafter, by the use of a common figure of speech, speak of them and other number symbols as number.

2. Elementary Number Notions. We know that $3+4=7$, because by counting 3 and then 4 more we reach 7. This may be seen by counting these groups,

● ● ●   ● ● ● ●

All the results of addition are primarily determined by counting. In practice, a number of simple addition results are determined by counting, and then these are made a matter of memory. $3 \times 4 = 12$, because by counting 3 groups of 4 each we reach 12. This is seen in the following arrangement:

● ● ● ●   ● ● ● ●   ● ● ● ●

The truth of a multiplication table is also established by counting. The number system shows that $3+4=4+3$;

for by counting 3 and then 4 more, we reach the same result as by counting 4 and then 3 more.

$$\bullet\ \bullet\ \bullet\quad \bullet\ \bullet\ \bullet\ \bullet = \bullet\ \bullet\ \bullet\ \bullet\quad \bullet\ \bullet\ \bullet$$

$3 \times 4 = 4 \times 3$ because 3 groups of 4 each make the same sum as 4 groups of 3 each.

$$\left.\begin{matrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{matrix}\right\} = \left\{\begin{matrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{matrix}\right.$$

Illustrations enough have been given to show that the integral number system is the real basis of the fundamental parts of arithmetic.

3. Fractions in the Number System. As long as no exact measurements are needed, nor accurate divisions attempted, the integral number system is sufficient. If a stick is more than 9 inches and less than 10 inches in length, we can not express its exact length by means of the integral number system. A similar difficulty arises in attempting to answer the question $8 \div 3 =$ what? To answer all such questions, *fractions* have been devised and made a part of the number system. The addition of fractions to the number system made possible many arithmetical operations which were before impossible. The field of arithmetic was thus greatly enlarged.

4. Incommensurables in the Number System. When the necessity for extracting roots arose in the development of arithmetic, it was found that many roots could not be exactly determined. For example, the square root of 2 lies between 1 and 2, between 1.4 and 1.5, between 1.41 and 1.42, between 1.414 and 1.415, etc. We may extend this

process of locating the square root of 2 between consecutive numbers of the number system as far as we please, but we can never find its exact value. Such numbers as the square root of 2, and the square and cube roots of other numbers which can not be exactly found, are called *incommensurable numbers* or merely *incommensurables.* Although such numbers can not be exactly expressed, the number system now includes them.

5. Numerical Arithmetic Complete. With the number system so developed as to include integers, fractions, and incommensurables, ordinary numerical arithmetic is complete. This means that in performing the operations of ordinary arithmetic no necessity arises for any other kind of numbers.

6. Literal Arithmetic. In percentage we frequently represent the base by b, the rate per cent by r, the percentage by p, the amount by a, and the difference by d. When we do this, we can transform the rules for the cases of percentage into the following forms:

(1) $p = b \times r.$

(2) $r = p \div b.$

(3) $b = p \div r.$

(4) $a = b + b \times r.$

(5) $d = b - b \times r.$

The symbols b, r, p, a, and d may be considered as particular numbers of the number system. When thought of in this way, they are mere abbreviations of numbers. Since they may be the abbreviations of any numbers whatsoever, we may think of the symbols themselves as

numbers. When a symbol, such as any of the above, is thought of in this way, it is called a *general number*. Such a number is frequently called a *literal number*. These symbols of general or literal numbers may have particular numerical values assigned to them. In order to find 8% of 250 we take form (1), on page 3, and put 250 instead of b, and .08 instead of r. We then have

$$p = b \times r = 250 \times .08 = 20.$$

7. Substitution. *The process of putting a particular number in the place of a general one is called substitution.*

By substitution all the results of general or literal arithmetic become particular. The solution of a problem in ordinary arithmetic is a mere matter of substituting particular numbers for general ones in the proper literal form, as is illustrated in the percentage problem of Section 6.

The area of a rectangle is the product of its length and width. If we represent area by a, length by l, and width by w, we at once have the general form

$$a = l \times w.$$

If we wish to find the area of a lot 66 feet long and 30 feet wide, we put 66 for l and 30 for w, and we have

$$a = l \times w = 66 \times 30 = 1980.$$

8. Algebraic Expression. *Any combination of literal numbers or of literal and arithmetical numbers by means of any or all of the signs of addition, subtraction, multiplication, division, involution, and evolution is an algebraic expression.*

$x + y - z$ is an algebraic expression and is read x plus y minus z. The algebraic expression $a \times b - c \div d + 4$ is

read a times b minus c divided by d plus 4. The word *function* is frequently used instead of *expression*. The parts of an algebraic expression separated by either of the signs + or − are called *terms*. In $ax + by - cyz$, there are three terms; viz., ax, by, and cyz.

9. Signs used in Algebraic Expressions. The signs $+$, $-$, $\times$, $\div$, and $\surd$ are used as in arithmetic. They denote addition, subtraction, multiplication, division, and root extraction, respectively. Multiplication is also indicated by a dot ($\cdot$), and by writing the characters adjacent to each other. $a \times b$, $a \cdot b$, and ab mean exactly the same thing; viz., a multiplied by b. Between arithmetical numbers also the signs $\times$ and $\cdot$ are used to denote multiplication. The multiplication of an arithmetical and literal number or of two literal numbers is denoted by writing them consecutively. $5b$ means $5 \times b$. ab means $a \times b$. $6ab = 6 \times a \times b = 6 \cdot a \cdot b$. The first form ($6ab$) is the only one in use and is read *six ab*.

There are five forms of division sign in general use, $6 \div 2$, $2\overline{)6}$, $6:2$, $\frac{6}{2}$, and $6/2$, all meaning exactly the same thing; viz., 6 divided by 2. In algebra the fourth and fifth forms are more frequently used than the others.

$\sqrt{x}$, $\sqrt[3]{a}$, $\sqrt[4]{b}$ are read square root of x, cube root of a, and fourth root of b, respectively.

a^2, b^3, x^4 are read a square, b cube, and x to the fourth power, respectively.

$$a^2 = aa; \quad b^3 = bbb; \quad x^4 = xxxx.$$

In the above expressions the 2, 3, and 4 are called **exponents**.

EXERCISES.

Read the following algebraic expressions:

1. $a+b-c+4.$ **2.** $4a-3b+ab.$

3. $ab-4ac+3ad-\frac{b}{c}.$

$\frac{b}{c}$ is read b divided by c, or b over c.

The last term of (3) should be read minus the quotient b divided by c, or minus the fraction b over c.

4. $2a\div b+5abc-4bc.$ **6.** $5a^3-4ax^2+\sqrt[3]{ab}.$

5. $x^2-4ax+\sqrt{b}.$ **7.** $6ab^4-\frac{4a^3}{3b^2}+\sqrt[4]{a^2x^3}.$

8. $5a\sqrt{x}+\frac{3x}{y}-7x^4.$

$5a\sqrt{x}$ is read $5a$ times the square root of x.

9. $xy^2z^3+8\frac{x}{y}-17\sqrt{x^2y^3}.$ **10.** $3x^4\sqrt[3]{yz}-\frac{3xy}{ab}+8a^3\sqrt[4]{x}.$

10. Precedence of Signs in Algebraic Expressions. *If only the signs + and − occur in an algebraic expression, the operations are to be performed in order from left to right.*

For example: $8-4+3+2-5=4.$

If only the signs × and ÷ occur in an algebraic expression, the operations are to be performed in order from left to right.

For example: $6\div 3\times 4\div 2=4.$

If the signs +, −, ×, and ÷ occur in an algebraic expression, the multiplications and divisions are first performed, and then the additions and subtractions.

For example: $4 + 3 \times 2 - 12 \div 6 \times 3 + 2 \times 4$.

Performing the multiplications and divisions in the above, we have $4 + 6 - 6 + 8$. Now performing the additions and subtraction, we get 12, which is the value of the expression.

11. Signs of Aggregation. The signs of aggregation or grouping are the parentheses ( ), braces { }, brackets [ ], and vinculum or bar ——. Each of these signs indicates that the expression within or under it must be treated as a whole. $(bc - ad) \div b$ means that the difference between bc and ad is to be divided by b. The expressions $(bc - ad) \div b$, $\{bc - ad\} \div b$, $[bc - ad] \div b$, and $\overline{bc - ad} \div b$ all mean precisely the same thing. The four signs of aggregation are all called by the general name *parentheses*. The different forms are necessary to avoid confusion when one or more groups are included within another group.

For example: $5 \times \{12 \div (7 \times \overline{6 + 2} \div 4 \div [2 \times 3 + 1])\}$.

This becomes $5 \times \{12 \div (7 \times 8 \div 4 \div 7)\}$, which in turn becomes $5 \times \{12 \div 2\}$, or $5 \times 6 = 30$.

Multiplication of a quantity within a parenthesis by any quantity is indicated by writing the multiplier before or after the parenthesis. $5(a + b)$ means $5 \times (a + b)$. $(a + b)5$ means $(a + b) \times 5$.

12. Coefficient. In the expressions $5a$, $3x$, and $7y$, 5, 3 and 7 are the coefficients of a, x, and y, respectively.

In an indicated product any factor or factors may be considered the *named part*, then all the other factors constitute the coefficient. Thus, in $8axy$, 8 is the coefficient of axy, $8a$ is the coefficient of xy, and $8ax$ is the coefficient

of y. In the first case axy is the named part, in the second xy, and the third y.

When no numerical coefficient precedes a literal expression, the coefficient 1 is understood.

Read the coefficients of y^2 in the following expressions:

$$8y^2,\ 5ay^2,\ 17axy^2,\ 11a^2bxy^2.$$

EXERCISES IN SUBSTITUTION.

Find the value of each of the following literal expressions, in which $a=4$, $b=2$, $c=3$, $d=5$, $x=6$, $y=1$, $z=10$:

1. $a+b-c+xy$.

Substituting the values given to the above letters, this expression becomes

$$4+2-3+6\times 1=6-3+6=9.$$

In a similar manner determine the values of the following expressions:

2. $a+x-y+3d$.

3. $ax+by+c$.

4. $x+y+a$.

5. ax^2+by^2+d.

6. $\frac{a}{b}+\frac{c}{d}+z$.

7. $x+y+z+d$.

8. x^2-y^2+a.

9. x^2+2x+1.

10. $x^2+2xy+y^2$.

11. $\sqrt{a}+yz+\frac{1}{d}$.

12. $\sqrt{6x}+\sqrt{y}$.

13. $5a+6b-4c$.

14. $3a-4b+c$.

15. $3x-4y+10$.

16. $(x+y)z+(y+z)x$.

17. $abc\div xyz$.

18. $ax-by+cz-3abc$.

19. $x^3+y^3+z^3-3xyz$.

20. $y+cx+(d-\sqrt{a})\div 4b$.

13. Use of General Number in Arithmetical Problems. All the problems of ordinary arithmetic may be made general by the use of general or literal numbers in place of the arithmetical numbers involved.

1. If John has 10 cents and Henry 12 cents, they together have $(10 + 12)$ cents. This becomes general by stating it thus: if John has a cents and Henry b cents, they together have $(a + b)$ cents.

2. If Mary is 10 years old and Susie is x years older, then Susie is $(10 + x)$ years old.

3. If a merchant sells a bushels of corn at b cents a bushel, and c bushels of wheat at d cents a bushel, and divides the money received equally among his e children, each one will receive $[(ab + cd) \div e]$ cents.

4. A man has a cents and b dimes. How many cents has he?

5. A man is a years old. His son is $\frac{1}{2}$ as old. What is the combined age of father and son?

6. I traveled b miles at c cents a mile, and d miles at e cents a mile. How far did I travel and what did it cost me?

7. A rectangle is x rods long and y rods wide. How many acres does it contain?

8. A farm a rods long and b rods wide is sold at c dollars per acre. Find the amount for which the farm sold.

9. How many hours will be required to travel x miles if one third the distance be traveled at a miles per hour and the remaining two thirds at b miles per hour?

10. A man has x dollars in cash; b men owe him each y dollars, c men owe him each z dollars. How much money would he have if his collections were made?

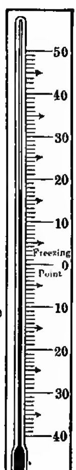
11. A rectangle is a rods long and b rods wide. Find the length of its diagonal.

12. How many hours will be required for a train running 30 miles per hour to travel x miles, if it makes n stops of b minutes each?

13. Find the cost of excavating a basement a feet long, b feet wide, and c feet deep at x cents per cubic yard.

14. A man sold from a flock of n sheep; the mth part of them for p dollars each. He then increased his flock by a sheep and sold the whole lot at x dollars per head. How much did he receive for the whole flock?

14. Opposite Numbers. On the scale of a thermometer, temperature is marked both ways from 0. On the centigrade thermometer, temperatures above freezing read from 0 *up*, and temperatures below freezing read from 0 *down*. Longitude is measured both east and west from a fixed 0 or prime meridian. Latitude is measured both north and south from the equator. We may consider direction along a line to the right or to the left. Rotation may be opposite to that of the hands of a clock, or it may be clockwise. Numbers which in some way indicate such opposites are called **Opposite Numbers.** The need of opposite numbers becomes apparent when we try to generalize the operations of arithmetic. $a - b$ indicates the subtraction of b from a.



Centigrade Thermometer

If $a = 10$ and $b = 9$, $a - b = 10 - 9 = 1$, which shows that 9 is 1 less than 10.

If $a = 10$ and $b = 10$, $a - b = 10 - 10 = 0$, which shows that 10 is 0 less than 10, or that 10 is equal to 10.

If $a = 10$ and $b = 11$, $a - b = 10 - 11$, which, whatever it may be, ought to show that 11 is 1 greater than 10.

If a man has \$500 and is in debt \$400, he is worth \$500 − \$400 = \$100.

If a man has \$500 and is in debt \$500, he is worth \$500 − \$500 = 0.

If a man has \$500 and is in debt \$600, he is worth \$500 − \$600. This statement harmonizes with the statements made in the other two cases. What does it mean? We may interpret it by saying that he owes \$100 more than he is worth, or that his liabilities exceed his assets by \$100, or that he is worth \$100 less than nothing.

15. Negative Number. Such questions as the above are answered by the extension of the number system so as to include *negative number*. We may think of the arithmetical number system as starting at 0 and extending indefinitely in a horizontal line to the right. It is an easy matter to think of a similar system extending indefinitely to the left from 0. These appear as follows:

200 ··· 100 ··· 60 ··· 50 ··· 4 3 2 1 0 1 2 3 4 ··· 50 ··· 60 ··· 100 ··· 200

$\longleftarrow$ − 0 + $\longrightarrow$

This extension doubles the scope of the number system. Integers, fractions, and incommensurables are all included in the extension to the left.

16. Positive and Negative. That part of the number system to the right of 0 is called *positive*. The positive character of a number is indicated by the use of a + sign before it. $+4$, $+a$, and $+x^2$ are positive numbers. In

practice the + sign is frequently omitted, so that the absence of a sign before a number shows that it is positive.

That part of the number system to the left of 0 is called *negative*. The negative character of a number is indicated by the use of a – sign before it. $-6a$, $-x^3$, and $-11x^2y$ are negative numbers.

The important idea in the two parts of the number system is that of *oppositeness*. When anything is represented by a positive number, its opposite is represented by a negative number. If time A.D. is positive, then time B.C. is negative. If distance to the right is positive, then distance to the left is negative.

ILLUSTRATIVE EXERCISES.

1. If two points are on the same meridian in latitude $+30°$ and $-20°$, respectively, how far apart are they? This means that one is in north latitude 30°, and the other in south latitude 20°. They are evidently $30° + 20° = 50°$ apart.

Draw a diagram illustrating this.

2. On a certain day the lowest temperature recorded was $-5°$ and the highest $+12°$. What was the difference in temperature between the lowest and highest?

3. A man was born in the year -31 and died in the year $+43$. How old was he?

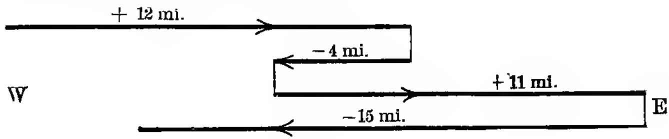
4. A man travels $+45$ miles from A, and his friend travels -80 miles from A. How far apart are they?

5. Two places are in $-63°$ and $+87°$ longitude, respectively. How far apart are they?

17. Extension of Meaning of Negative. In the above exercises and illustrations we have thought of negative number as beginning at 0 and extending in the opposite direction from that of positive number, which also begins

at 0. Beginning at zero is not a necessary part of the meaning. The necessary part is that of *oppositeness.* Number starting anywhere is *negative* if it denotes extension in the opposite direction to that of positive number.

A man travels east 12 miles, then west 4 miles, then east 11 miles, then west 15 miles. If we select east as the positive direction, then west is the negative direction. We may then say that a man travels $+12$ miles, then -4 miles, then $+11$ miles, and then -15 miles; as shown in the following diagram.



18. Double Use of the Signs + and −. The signs + and − in algebra retain their arithmetical sense, and denote addition and subtraction, respectively. When used in this sense they are called *signs of operation.* The signs + and − are also used to denote the quality of a number, that is, to indicate that the number belongs to the positive or negative part of the number system. In this sense, the signs + and − denote oppositeness, and are called *signs of quality.* Smaller signs slightly elevated are sometimes placed before a number to denote its quality.

For example: ${}^{+}5$, ${}^{-}5$, ${}^{+}a$, ${}^{-}a$, ${}^{+}15\,b$, ${}^{-}18\,c$.

19. Algebraic Number. When the quality of a number is considered, the number becomes *algebraic.* The signs of algebraic number are + and −. The *absolute value* of a number is its value without regard to quality. $+a$, $+b$, $+11$, -7, are *algebraic* numbers. a, b, 11, 7, are their *absolute* values.

CHAPTER II.

DEFINITIONS.

20. Identity. In the equality $5x - 2x = 3x$ we have merely the statement that $5x$ diminished by $2x$ becomes $3x$. No question need be asked concerning the *number* or *value* represented by x. The equality is true, whatever value x may have. If $x = 1$, the equality becomes $5 \times 1 - 2 \times 1 = 3 \times 1$. If $x = 5$, it becomes $5 \times 5 - 2 \times 5 = 3 \times 5$. If x be any number a, it becomes $5a - 2a = 3a$.

An equality, true for all values of the letters considered, is called an identity.

In an identity both sides of the equality may be reduced to the same form. $x^2 + 2ax + y^2 - 2ax = x^2 + y^2$ is an identity because the left side becomes $x^2 + y^2$ by uniting $2ax$ and $-2ax$.

21. Equation. The equality $x + 3 = 7$ is entirely different from that considered in the preceding Section. If x be 1, the equality does not exist, for $1 + 3$ is not equal to 7. Neither does it exist if x be 3, for $3 + 3$ is not 7. If x be 4, the equality exists, for then we have $4 + 3 = 7$, a numerical identity. We see that the equality $x + 3 = 7$ restricts the value of x to 4.

An equality which contains one or more restricted letters is called an equation.

An identity is usually distinguished from an equation by having its sign of equality written thus, $\equiv$, while the equation retains the sign $=$. The sign $\equiv$ is read "is identical with" or "identically equals." The sign $=$ is read "is equal to" or "equals."

$5a + 3a \equiv 8a$ is read $5a$ plus $3a$ is identical with $8a$.

$3x + 4 = 20$ is read $3x$ plus 4 equals 20.

22. Variables. *The letters of an equality which are restricted in value are called variables.*

They are generally, although not necessarily, represented by the last letters of the alphabet.

In the equality $x + y = 5$, x and y are variables.

23. Constants. *The letters of an equality which are not restricted and all arithmetical numbers are called constants.*

They are generally, although not necessarily, represented by the first letters of the alphabet.

In the equality $ax + by = c$, the constants are a, b, and c.

EXERCISES.

Distinguish between *equation* and *identity* in the following equalities; also point out the variables and constants.

1. $5x - 4x + x = 2x$.
2. $8y + x - 4y + 6x = 4y + 7x$.
3. $3x - 5 = 10$.
4. $12a - 3a + 4b = 9a + 4b$.
5. $4a + 10 = 18$.
6. $2ax + 5ax - 3ax = 4ax$.
7. $5y - 2y = 15$.
8. $4xy + 6xy - 2xy = 8xy$.
9. $x^2 - 2ax + a^2 + 2ax = a^2 + x^2$.
10. $8z - 5z = 12$.
11. $ax + by = c$.
12. $3x + 4x = 5x + 2x$.
13. $7x + a = 12$.
14. $a^2x + 5a^2x = a^2x(8 - 2)$.
15. $5x + 9x - 7x = 28$.

24. The Root of an Equation. *A value of the variable which, when substituted for the variable, reduces an equation to an identity, is called a root of the equation.*

The equation $x - 5 = 8$ asks what number diminished by 5 equals 8. The answer, $x = 13$, is the root, for $13 - 5 = 8$ is an identity.

The process of obtaining the value of the root is called solving an equation.

25. Axioms. In solving an equation certain elementary facts are taken for granted; that is, their truths are accepted without proof. Such self-evident truths are called *axioms.* The three following are of use to us at present:

(1) *Numbers equal to the same number, or to equal numbers, are equal to each other.*

Ex. $4 + 2 = 6$, $3 + 3 = 6$; hence $4 + 2 = 3 + 3$.

If $3a - 5 = 10$, and $2b + 4 = 10$, then $3a - 5 = 2b + 4$.

(2) *If equals be added to or subtracted from equals, the results are equal.*

Ex. $3 + 2 = 5$; then $3 + 2 - 2 = 5 - 2$.

If $a + b = c$, then $a + b - b = c - b$.

If $x + y = 5$, then $x + y + 4 = 5 + 4$.

(3) *If equals be multiplied or divided by equals, the results are equal.*

Ex. $\frac{10}{3} = 3 + \frac{1}{3}$.

Multiply by 3, $\frac{10}{3} \times 3 = 3 \times 3 + \frac{1}{3} \times 3$, or $10 = 9 + 1$.

If $4x = 20$, then $\frac{4x}{4} = \frac{20}{4}$, or $x = 5$.

26. Solution of Exercises. The algebraic equation can be used to advantage in the solution of many exercises found in ordinary arithmetic. The process consists in first expressing the exercise as an algebraic equation, and then applying the axioms so as to find the root. This will be illustrated in the following exercises.

EXERCISES.

1. A and B have \$900; A has \$100 more than twice what B has. How much has each?

SOLUTION.

Let $x =$ B's money.

$2x + \$100 =$ A's money.

$x + 2x + \$100 =$ both A's and B's money.

$\$900 =$ both A's and B's money.

By Axiom (1), $x + 2x + 100 = 900.$

By Axiom (2), $x + 2x = 900 - 100 = 800$ (subtracting 100 from equals).

$3x = 800.$

By Axiom (3), $x = 266\frac{2}{3}$, B's money.

$2x + 100 = 533\frac{1}{3} + 100 = 633\frac{1}{3}$, A's money.

2. A and B have \$1500; A has \$300 less than 3 times B's. How much has each?

3. What number added to twice itself will make 900?

4. The sum of two numbers is 84; the larger is 11 times the smaller. What are the numbers?

5. John has \$300 more than Henry; they both have \$2100. How much has each?

6. A house and lot cost \$3700; if the house is worth \$700 more than the lot, what is the value of each?

7. Divide $720 among three men so that the second shall have twice as much as the first, and the third 3 times as much as the first.

8. Divide 440 into three parts so that the second part shall be 100 more than the first, and the third part as much as the sum of the first and second parts.

9. A horse and carriage cost $288; the carriage cost $\frac{4}{5}$ as much as the horse. How much did each cost?

SOLUTION.

Let $x =$ cost of horse.

$\frac{4}{5}x =$ cost of carriage.

$x + \frac{4}{5}x =$ cost of both.

$\$288 =$ cost of both.

By Axiom (1), $x + \frac{4}{5}x = 288$.

By Axiom (2), $5x + 4x = 288 \times 5$, multiplying by 5.

$9x = 1440$.

By Axiom (3), $x = 1440 \div 9 = 160$, cost of horse.

$\frac{4}{5}x = \frac{4}{5}$ of $160 = 144$, cost of carriage.

10. What number increased by $\frac{2}{3}$ of itself is 550?

11. One third of a number increased by $\frac{1}{4}$ of the number is 455. What is the number?

12. A's money is $\frac{2}{3}$ of B's money; together they have $1300. How much has each?

13. Four times a number increased by $\frac{3}{4}$ of the number is 475. What is the number?

14. Two numbers added together make 80; the greater is 5 more than 4 times the lesser. What are the numbers?

15. If to my age you add its half and its third and 50 years more, the sum will be 3 times my age. What is my age?

16. If to the double of a number you add its half and 42 more, the sum will be 4 times the number. What is the number?

17. One number is $\frac{5}{8}$ of another; their sum is 156. What are the numbers?

18. $7x - 5x + 11x = 390$. Find x.

19. Divide the number 99 into three parts so that the first shall be 2 times the second and 3 times the third.

20. A man bought two houses for \$4400, paying 10 times as much for one as for the other. What did each cost?

21. A man paid \$700 more for one house than for another; the cost of one being $\frac{4}{5}$ of the cost of the other. What was the cost of each?

22. One seventh of a number exceeds $\frac{1}{8}$ of it by 560. What is the number?

23. What number added to $\frac{2}{3}$ of itself will make 1000?

24. What number diminished by $\frac{4}{5}$ of itself will make 60?

25. What number increased by $\frac{1}{2}$ and $\frac{1}{3}$ of itself will make 110?

CHAPTER III.

ADDITION AND SUBTRACTION.

27. Arithmetical Addition. In elementary arithmetic, to add two numbers, 4 and 5 for example, is to find in the number system a number 9 by the process of counting, first 4 and then 5 more. The number so found is called the *sum*, and the numbers added are called the *addends*. The addition of any number of addends furnishes only an extension of the above process of continuous counting. If we think of number represented as heretofore upon a scale, then addition may be represented as follows :

Let us add 5, 3, and 6.

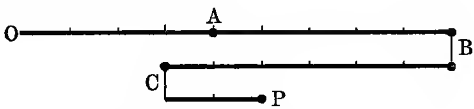
First, we count 5 from O, which takes us to A; then 3 more, which makes 8 and takes us to B; and finally 6 more, which makes 14 and takes us to P. In this illustration O is the zero point from which counting proceeds, and P is the *terminal point*. Hence, the sum represents the counted distance of the terminal point from zero. In practical addition elementary sums are remembered, and thus the actual counting is avoided.

28. Algebraic Addition. We have already seen that ordinary algebra contains not only positive number (forward counting), but also negative number (backward

counting). Part or all of the addends may therefore be negative. Hence, the definition of algebraic addition must be extended accordingly.

Algebraic addition is the process of finding a number (sum) in the algebraic number system represented by the terminal point reached by the successive forward and backward countings indicated by the addends.

Thus to add 4, 5, -6, and 2 is to count 4, then 5 more, giving 9, then backward 6 to 3, then forward 2 to 5; 5 is the terminal point of the successive countings and is the sum of the given addends.



On the diagram the counting is from O to A, A to B, B to C, C to P.

To add 4, -6, -3, and $+2$ is to count 4, then backward 6 to -2, then backward 3 more to -5, and then forward 2 to -3. The terminal point of the successive countings is -3, which is therefore the sum of the addends given. Show this by a diagram.

EXERCISES.

Find as above the sum in each of the following, and illustrate by a diagram:

1. $4, 5, -3$.
2. $6, -7, -3$.
3. $-4, 5, -1$.
4. $8, -9, +6, -4$.
5. $-7, -3, -1, +5$.
6. $-8, -9, -4$.
7. $4, -8, +6, -5$.
8. $-6, +9, -10, +8$.
9. $-4, -2, +8, +6$.
10. $5, +6, -7, -10$.

29. Monomials and Like Monomials. *Algebraic expressions consisting of single terms are called monomials.*

Monomials containing the same literal parts, each literal part having the same exponent, are called like monomials.

$6x$, $-4ab$, $3xy$, are monomials. $4a$, $2a$, and $-3a$ are like monomials. To add $4a$, $2a$, and $-3a$, is to count $4a$, and then $2a$ to $6a$, and then backward $3a$ to $3a$. $3a$ is the terminal point of the successive countings, and is therefore the sum.

EXERCISES.

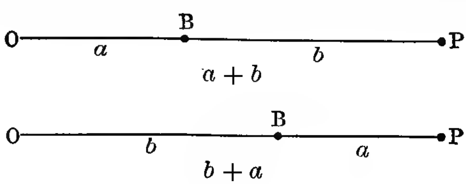
Add the following monomials:

1. $4b,\ 5b,\ -2b,\ -3b.$
2. $3a^2b,\ -6a^2b,\ -5a^2b.$
3. $-5xy,\ 7xy,\ -2xy.$
4. $6ax^2,\ 3ax^2,\ -10ax^2+ax^2.$
5. $11abc,\ -10abc,\ -4abc.$
6. $8x^2,\ -5x^2,\ +7x^2,\ -0x^2.$
7. $-2x^2y,\ +4x^2y,\ x^2y,\ -5x^2y.$
8. $9axy, -5axy, -7axy, 4axy.$
9. $3\sqrt{xy},\ -2\sqrt{xy},\ +6\sqrt{xy},\ -\sqrt{xy}.$
10. $4a\sqrt{z},\ +5a\sqrt{z},\ -9a\sqrt{z}.$

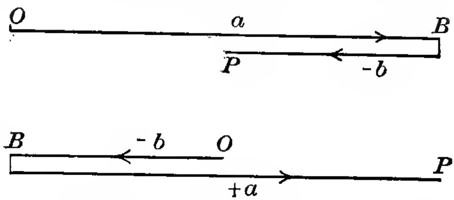
30. The Commutative Law. We know that in arithmetic $3+4=4+3$. This is known as the *Commutative Law*, and means that the addends may be taken in any order. The law is applicable to any number of addends. This law holds in the extended number system of algebra. It is algebraically stated as follows:

$$a+b=b+a.$$

The following diagrams show the truth of this law:



That $a - b = -b + a$ is shown as follows:



The distance from O to P is the same in each case.

31. The Associative Law. We know that in ordinary arithmetic, in adding 2, 3, 4, 5, we may, if we wish, first add 3 and 4, and add their sum to 2, and then add 5. We may, in fact, associate the addends in any manner we choose. This is known as the *Associative Law*. This law holds in the extended number system of algebra. It is algebraically stated as follows:

$$a + b + c = a + (b + c).$$

This law, like the Commutative Law, applies to both positive and negative numbers.

32. Addition of Monomials. Let us find the value of the expression

$$3x - 2x + 4x - 7x + 8x.$$

By applying the Commutative Law we may write this expression in the equivalent form $3x + 4x + 8x - 2x - 7x$. Now, by the Associative Law we may add all the positive numbers into the one sum of $15x$, and all the negative numbers into another sum of $-9x$, and thus write the expression in the equivalent form $15x - 9x$, which we at

once know to be $6x$. All this may be shown in the following scheme:

$$
\begin{aligned}
&3x-2x+4x-7x+8x \\
&=3x+4x+8x-2x-7x, && \text{by Commutative Law.} \\
&=15x-9x, && \text{by Associative Law.} \\
&=6x, && \text{by adding.}
\end{aligned}
$$

The sum of a positive and negative number is equal to the difference of their absolute values with the sign of the greater prefixed; *e.g.* $6-4=+2$, while $9-13=-4$.

RULE. *To add like monomials, add the positive terms, then add the negative terms; to the difference of the absolute values of the two sums prefix the sign of the greater.*

EXERCISES.

1. Add $3x,\ -5x,\ -9x,\ +11x,\ -3x,\ +7x$.

We may for convenience arrange the solution thus:

$$
\begin{array}{rr}
+\ 3x & -\ 5x \\
11x & -\ 9x \\
\underline{7x} & \underline{-\ 3x} \\
21x & -17x \\
\underline{-17x} & \\
+\ 4x &
\end{array}
$$

It will be noticed that we have arranged in the first column all the positive numbers, and in the second all the negative numbers. This is a mere matter of convenience. The student should early accustom himself to pick out and unite the positive numbers mentally, and likewise the negative numbers, merely writing down the results. Unless the coefficients are very large, a little practice will enable the student to do this with accuracy and rapidity.

2. Add $5\,xyz,\ -11\,xyz,\ 18\,xyz,\ -4\,xyz,\ -13\,xyz.$

3. Add $15\,a^2bc,\ 10\,a^2bc,\ -2\,a^2bc,\ -18\,a^2bc.$

4. Add $10\,x^2,\ -5\,x^2,\ -2\,x^2,\ 16\,x^2,\ -8\,x^2.$

5. Add $\frac{3}{5}\,ax,\ \frac{4}{5}\,ax,\ -\frac{7}{5}\,ax,\ -\frac{1}{5}\,ax,\ -\frac{2}{5}\,ax.$

6. Add $3\sqrt{2\,x},\ -5\sqrt{2\,x},\ -3\sqrt{2\,x},\ -\sqrt{2\,x},\ 6\sqrt{2\,x}.$

7. Add $4(a+b),\ -3(a+b),\ -4(a+b),\ 2(a+b).$

Algebraic expressions containing like quantities within a parenthesis may be added as monomials. Thus, in Exercise 7, the quantity $a+b$ is common to each addend, hence the sum is $-(a+b)$.

8. Add $5(x^2+y^2),\ -3(x^2+y^2),\ 7(x^2+y^2),\ -8(x^2+y^2).$

9. Add $\frac{1}{2}(ax+by+2),\ -\frac{3}{2}(ax+by+2),\ (ax+by+2).$

10. Add $\frac{5}{8}(x^2-2\,y),\ -\frac{3}{8}(x^2-2\,y),\ -\frac{1}{8}(x^2-2\,y).$

11. Add $3(\sqrt{x}+\sqrt{y}),\ -8(\sqrt{x}+\sqrt{y}),\ 6(\sqrt{x}+\sqrt{y}).$

12. Add $4\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right),\ -3\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right),\ -2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right).$

33. Addition of Polynomials. *An algebraic expression consisting of more than one term is a polynomial.*

If it has two terms, it is called a binomial; and if three, a trinomial.

If we desire the sum of $4\,a+b$ and $3\,a-4\,b$, we may indicate that sum thus: $4\,a+b+3\,a-4\,b$.

Now, by the Commutative Law, we may write this $4\,a+3\,a+b-4\,b$, which by the Associative Law becomes $7\,a-3\,b$.

However many polynomials we might have to add, the process would be an extension of the above. We have then the following rule for adding polynomials:

RULE. *Unite the like terms of the various polynomials into sums, and connect these by their proper signs to form the polynomial sum.*

EXERCISES.

1. Add $2\,ax + 3\,by + 5\,cz$, $4\,ax - 5\,by$, $7\,by - 3\,cz$, $11\,ax - 4\,by - 6\,cz$.

For convenience we may arrange these polynomials thus:

$$\begin{array}{rrr}
2\,ax & +\,3\,by & +\,5\,cz \\
4\,ax & -\,5\,by & \\
 & 7\,by & -\,3\,cz \\
11\,ax & -\,4\,by & -\,6\,cz \\
\hline
17\,ax & +\quad by & -\,4\,cz
\end{array}$$

It should be noticed that in the arrangement like terms have been placed in columns.

2. Add $5x+3y+4z$, $-3x-5y+z$, $7x-2y-z$, $10x+y-2z$.

3. Add $5\,ax-6\,by+cz$, $3\,ax+by+3\,cz$, $-4\,ax+3\,by-3\,cz$, $3\,ax-3\,by+3\,cz$.

4. Add $4\,x^2+3\,y^2-4$, $-3\,x^2+2\,y^2-6$, x^2-y^2+7.

5. Add $3\,x^2+4\,y^2+3\,z^2$, $-x^2-4\,y^2+2\,z^2$, $11\,x^2-y^2-z^2$.

6. Add $4\,xy-3\,yz+7\,zx$, $4\,yz-7\,xy$, $3\,yz+4\,zx$, $-4\,xy+3\,zx-2\,yz$, $-xy-zx$.

7. Add $4\sqrt{xy}+3\sqrt{yz}+\sqrt{zx}$, $-2\sqrt{yz}-3\sqrt{zx}$, $-2\sqrt{xy}-3\sqrt{yz}+\sqrt{zx}$.

8. Add $3\,abc-4\,xyz+7\,lmn$, $-4\,lmn+12\,xyz$, $5\,xyz-4\,abc+3\,lmn$, $-5\,abc-7\,lmn$.

34. Identity. *The sum of the addends is identically equal to the sum.*

Thus, $$3\,x+4\,x-5\,x\equiv 2\,x.$$

Also, $$2\,a+b+5\,a-4\,b-6\,a+4\,b\equiv a+b.$$

Since an identity is true for all values of the letters involved, we may make use of the identity existing between addends and sum to

verify our results in addition. If in the first illustration above we make $x = 1$, it becomes

$$3 + 4 - 5 \equiv 2, \text{ or } 2 \equiv 2.$$

If in the second illustration we put $a = 1$ and $b = 1$, it becomes

$$2 + 1 + 5 - 4 - 6 + 4 \equiv 1 + 1, \text{ or } 2 \equiv 2.$$

The use of 1 for each of the letters is convenient, but not necessary. We may use any value whatever. Thus, in the second illustration above, we may put

$$a = 3 \text{ and } b = 5.$$

The identity then becomes

$$6 + 5 + 15 - 20 - 18 + 20 \equiv 3 + 5, \text{ or } 8 \equiv 8.$$

EXERCISES.

Add the following quantities, and verify the results by substituting particular values for the letters used:

1. $3a + 4b,\ 5a - 6b,\ 7a - 4c,\ 5b + 11c.$

2. $4x - 3y,\ 5x + 3y,\ -7x + y,\ 3x - 2y.$

3. $2a^2 - 3b^2,\ 5b^2 + 4a^2,\ 7a^2 - 5b^2,\ -3a^2 - b^2.$

4. $5x^2 + 4xy + 3y^2,\ 2x^2 - 5xy + 6y^2,\ 3x^2 - 8y^2,\ 8xy.$

5. $-3ax + 4by + c,\ 5ax - 6by - 3c,\ ax - 3by + 2c.$

6. $12x - 3y + z,\ 6x - 4y + 7z,\ -8x - 3y + 3z.$

7. $x^2 + y^2 + z^2,\ 3x^2 + 4y^2 + 5z^2,\ -4x^2 - 8y^2 + z^2.$

8. $x^2 - 3y,\ 5x + 4y + 3,\ x^2 + 4y,\ y^2 - 3x.$

9. $3\sqrt{x} - 4\sqrt{y},\ 5\sqrt{x} + 8\sqrt{y},\ -6\sqrt{x} + \sqrt{y}.$

10. $4a^{\frac{1}{2}} - 5b^{\frac{1}{3}} + 6b,\ 3a^{\frac{1}{2}} + 2b^{\frac{1}{3}} - 4b.$

NOTE. $a^{\frac{1}{2}}$ means $\sqrt{a}$; $b^{\frac{1}{3}}$ means $\sqrt[3]{b}$.

11. $5\sqrt[3]{a} - 6\sqrt[3]{b} + c,\ -3\sqrt[3]{a} + 4\sqrt[3]{b} - 4c,\ 2\sqrt[3]{a} - 3\sqrt[3]{b} + 2c,\ 5\sqrt[3]{a} - 3\sqrt[3]{b} + 3c.$

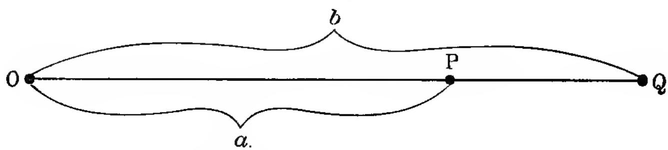
12. $3axy + 4byz + 6czx,\ 2byz - 5axy + 6czx,\ -4axy + 2byz - 7czx,\ 2axy - byz - 7czx.$

35. Subtraction. *Subtraction is the process of finding one addend, when the other addend and the sum of the two addends are given.*

The given addend is called the *subtrahend*, the given sum the *minuend*, and the addend which is to be found the *remainder* or *difference*.

Referred to the number system, we may say that to subtract a from b is to find the amount and direction of counting necessary to pass from a to b, both a and b beginning at the zero point.

This is shown on our number system diagram as follows:



If P is the terminal point of the subtrahend a and Q that of the minuend b, then the distance and direction from P to Q is the result of subtracting a from b.

36. Cases of Subtraction. A consideration of the definition of subtraction gives us the following four cases:

(1) If we subtract a positive number from a positive number, the remainder is the arithmetical difference of the absolute values, positive or negative, according as the absolute value of the subtrahend is less or greater than the absolute value of the minuend.

For example:

$$7a-(+3a)=4a,\ 7a-(+10a)=-3a.$$

(2) If we subtract a negative number from a positive number, the remainder is the positive arithmetical sum of the absolute values.

For example: $5x-(-3x)=8x.$

(3) If we subtract a positive number from a negative number, the remainder is the negative arithmetical sum of the absolute values.

For example: $-6ab-(+4ab)=-10ab.$

(4) If we subtract a negative number from a negative number, the remainder is the positive or negative arithmetical difference of the absolute values according as the absolute value of the subtrahend is greater or less than the absolute value of the minuend.

For example:

$$-7x-(-11x)=4x, \quad -13x-(-9x)=-4x.$$

The student should verify the truth of these cases by testing them on the number system diagram.

EXERCISES.

1. From $19x^2y$ subtract $11x^2y$.
2. From $7ab$ subtract $-5ab$.
3. From $-9xyz$ subtract $12xyz$.
4. From $-17x$ subtract $-13x$.
5. From $-14x^2$ subtract $-23x^2$.
6. From $27a^3y$ subtract $43a^3y$.
7. From $15y^2$ subtract $15y^2$.
8. From $-7abx^3$ subtract $-7abx^3$.
9. From $17\sqrt{xy}$ subtract $-12\sqrt{xy}$.

10. From $11(a+b)$ subtract $15(a+b)$.

11. From $-6(x+y)$ subtract $9(x+y)$.

12. From $-11(a^2+b^2)$ subtract $-7(a^2+b^2)$.

13. From $18(xy+yz+zx)$ subtract $-7(xy+yz+zx)$.

14. From $23(ax+by+c)$ subtract $40(ax+by+c)$.

15. From $-16(y^2+4ax)$ subtract $-21(y^2+4ax)$.

37. Subtraction of Monomials and Polynomials. In the cases considered in Section 36 it may be noticed that the subtraction of any number is equivalent to the adding of an equal opposite number.

Illustrations:

$$\begin{aligned} 7a-(+4a) &\equiv 7a+(-4a)=3a. \\ 7a-(-4a) &\equiv 7a+(+4a)=11a. \\ -7a-(-4a) &= -7a+(+4a)=-3a. \\ -7a-(+4a) &= -7a+(-4a)=-11a. \end{aligned}$$

Hence, we have the following rule:

RULE. *To subtract one number from another number, change the sign of the subtrahend and proceed as in addition.*

For example: The problem, *from* $18ax^2y$ *take* $-5ax^2y$ is equivalent to this problem, *to* $18ax^2y$ *add* $5ax^2y$.

In practice the change in sign should always be made mentally.

RULE. *To subtract one polynomial from another, change the sign of each term of the subtrahend and proceed as in addition.*

EXERCISES.

1.

$$\begin{array}{lr} \text{From} & 9x^2+3xy-11y^2+7 \\ \text{Subtract} & \underline{7x^2-5xy-17y^2+9} \\ & 2x^2+8xy+6y^2-2 \end{array}$$

In this we think of the $7x^2$ as negative, and add it to $9x^2$, giving $2x^2$. The $-5xy$ is thought of as positive, and added to $3xy$, giving $+8xy$, and so on with the other terms of the subtrahend.

2. From $9x^3-11x^2+5y$ subtract $14x^3-7x^2-8y$.

3. From $11a^4-15b^4-13a^2b^2$ subtract $7a^4+10b^4-3a^2b^2$.

4. From $4xy+3x^2-11y^2$ subtract $7xy-4x^2-13y^2$.

5. From $18ax-14by+11c$ subtract $3ax+19by-5c$.

6. From $5\sqrt{x}+8\sqrt{y}-13a$ subtract $8\sqrt{x}-5\sqrt{y}+3a$.

7. From $19x^3+7x-5$ subtract $4x^3+11x^2-8$.

Arranging for subtraction, we have

$$\begin{array}{rrrr} 19x^3 & & +7x & -5 \\ 4x^3 & +11x^2 & & -8 \\ \hline 15x^3 & -11x^2 & +7x & +3 \end{array}$$

In the minuend there is no term in x^2, or we may say there is the term $0x^2$. Hence the $11x^2$ is to be subtracted from $0x^2$, and of course gives a remainder of $-11x^2$. In the subtrahend there is no term in x, so there is nothing to subtract from $7x$, or the remainder is $7x$.

8. From $5x^4-3x^3+3x+6$ subtract $6x^4-3x^3+x^2-5$.

9. From $4x^2y^2+6xy^3-3x^3y+y^4$ subtract $3y^4+4xy^3-6x^2y^2$.

10. From $3(a+b)-5(x+y)+6c$ subtract $6(x+y)-5(a+b)-5$.

11. From $14\sqrt{xy}+13y^3-16x$ subtract $15y^3+17y-12\sqrt{xy}$.

12. From $4ax+5y^2-6xz+13$ subtract $21+17xz-15ax+11y$.

13. From $18y^3-27z^3+42yz$ subtract $18yz-36y^3+17z^3$.

14. From $36a^3-27a^2b-17ab^2+b^3$ subtract $13b^3+17ab^2-37a^2b$.

15. From $13x^2+17xy-5$ subtract $18y^2-17yz+32$.

38. Identity. Subtraction may be expressed as

$$\text{Minuend} - \text{Subtrahend} \equiv \text{Remainder}.$$

This is an identity and will, therefore, be true for any values of the letters involved. This identical relation furnishes a convenient method for verifying the results of subtraction.

For example:

$$\begin{array}{rl} 7\,ab - 14\,a^2 + 11\,b^2 & \text{Minuend.} \\ 4\,ab - 11\,a^2 + 19\,b^2 & \text{Subtrahend.} \\ \hline 3\,ab - 3\,a^2 - 8\,b^2 & \text{Remainder.} \end{array}$$

If we put $a = 1$, $b = 1$, we get

$$\begin{array}{r} 7 - 14 + 11 \equiv + 4 \\ 4 - 11 + 19 \equiv + 12 \\ \hline 3 - 3 - 8 \equiv - 8 \end{array}$$

which shows our result to be true.

EXERCISES.

Perform the following subtractions and verify by putting the letters each equal to 1:

1. From $9\,a^2 + 11\,ab - 13\,a$ take $8\,a^2 + 14\,ab - 9\,a$.
2. From $3\,x - 4\,y + 7$ take $x - 2\,y + 8$.
3. From $4\,x^2 + 5\,y^2 + 7\,xy$ take $5\,x^2 - 4\,xy + 3\,y^2$.
4. From $14\,ax + 12\,by + 7$ take $-3\,ax + 2\,by - 5$.
5. From $4(x + y) + 7\,z - 4$ take $3(x + y) - 5\,z - 8$.
6. From $16(x^2 + xy + y^2) + z^2 - w^2$ take $12(x^2 + xy + y^2) - 4\,z^2 + 3\,w^2$.
7. From $2(b^2 - 4\,ac) + x^2 + y^2$ take $5(b^2 - 4\,ac) + 5\,x^2 - 6\,y^2$.
8. From $ax + by + c$ take $a'x + b'y + c'$.

9. From $ax^2 + 2\,hxy + by^2 + 2\,gx + 2\,fy$ take $a'x^2 + 2\,h'xy + b'y^2 + 2\,g'x + 2\,f'y + c'$.

10. From $x^2 + y^2 + ax + by$ take $3\,x^2 + 3\,y^2 + lx + my$.

11. From $a\sqrt{x} + b\sqrt{y} + c$ take $a'\sqrt{x} + b'\sqrt{y} + c'$.

12. From $5\sqrt{x^2+y^2} + 7\sqrt{y^2+z^2}$ take $3\sqrt{x^2+y^2} + 8\sqrt{y^2+z^2}$.

39. Removal of Parentheses. If we recall the principles of addition and subtraction already developed, we can by means of them remove parentheses preceded by the + or − signs.

For $\qquad a + (b - c + d) = a + b - c + d,$

and $\qquad a - (b - c + d) = a - b + c - d.$

Hence, *a parenthesis preceded by a + sign may be omitted without any change in the signs of the terms inclosed. A parenthesis preceded by a − sign may be omitted if the sign of each term within it is changed.*

For example:

$$ax + (3\,ab - 4\,cy - 3\,az) = ax + 3\,ab - 4\,cy - 3\,az.$$
$$ax - (3\,ab - 4\,cy - 3\,az) = ax - 4\,ab + 4\,cy + 3\,az.$$

EXERCISES.

Remove the parentheses in the following and unite like terms:

1. $3\,a - (2\,a + 4\,b - c)$. **2.** $3\,x - 4\,y - (2\,x + a - 3\,y)$.

3. $5\,a^2 - (b + 3\,a^2) - 4\,b - (2\,a^2 - 3\,b)$.

4. $3\,ax - [a^2 - 3\,ax + xy - (xy - 5\,ax + a^2)]$.

Remove the [ ] first, and we have

$$3\,ax - a^2 + 3\,ax - xy + (xy - 5\,ax + a^2).$$

Now since the parenthesis is preceded by a + sign it may be omitted, and we have, when we unite the like terms, ax as the result.

5. $2a - 3b - [7a - (4b - \overline{5a - b} + 2b)]$.

6. $3ax - 5by - [7by + 3ax - (5ax - 3by)]$.

7. $11x - (7y - 3x) + 5y - [3x - (4y + 5x)]$.

8. $5x - \{7a - [3x - (5a - \overline{2x - a})]\}$.

9. $17a - 12x - \{15a - 11x + (3a - 2x) - \overline{5x - 4a}\}$.

10. $2x - \{6y + [6z - (x - y + z) - 2x] + 3y\}$.

In exercise 10 and the following put $x = 5$, $y = 4$, and $z = 2$, and find the value of the expressions.

11. $5y - 3z + [x - y + 2z - (3x + \overline{2y - 4z})]$.

12. $3z - \{2x - [5z - (4y - \overline{7x - 2y} + z) - 3x] + 2y\}$.

13. $2x + 3y - (4y - z + x) - [5x - (7y + 3x)]$.

14. $xy - (yz - zx - \overline{2xy - 2yz})$.

15. $3xy - \{4x^2 - y^2 - [2xy + 3x^2 - 5y^2] - (3x^2 - 4y^2)\}$.

40. Insertion of Parentheses. Many times it is just as important to insert parentheses in an algebraical expression as it is to remove them. Evidently any number of terms may be inclosed in a parenthesis without change if the parenthesis is preceded by a + sign. Any number of terms may be inclosed in a parenthesis preceded by a − sign, if the sign of every term so inclosed is changed.

For example:

$$ax + by - cz + ab = ax + (by - cz + ab),$$
$$ax + by - cz + ab = ax - (-by + cz - ab).$$

The number in the parenthesis is thought of as one quantity, and hence may be considered as one term. This gives an extension in meaning to the word *term*. In the illustrations just given the expressions on the left have four terms, but those on the right have only two terms.

EXERCISES.

1. $3a^2 + 4ab - 3c + d$. Inclose the last three terms in a parenthesis preceded by $-$.

2. $ay + by - cy - xy$. Inclose the last three terms in a parenthesis preceded by $-$.

In each of the next four exercises inclose all the terms containing x in a parenthesis preceded by $-$.

3. $5 - 3x + ax - 4abx$.

4. $7y + 3z + zx - 3x - 5yx$.

5. $13 - 5ax + 11x - 16ax + 27a$.

6. $27y - 30y + 22xy - 13bx + 21x - 30a + 16b$.

In the following exercises put all the terms containing a in a parenthesis preceded by $-$, and all the terms containing y in a parenthesis preceded by $-$.

7. $15 - 3a + 4y - 5ax + 7xy + 3ab - 5by$.

Rearranging, we have

$$15 - 3a - 5ax + 3ab + 4y + 7xy - 5by.$$

Then $\quad 15 - (3a + 5ax - 3ab) - (-4y - 7xy + 5by)$.

8. $25x + 5y - 3x + 7ax - 11by + cx - 3ac + 11$.

9. $b^2 - ab + cy - 2ac^2 - 4b^2y + 11a - 4y$.

10. $3x + 2a - 4y + 5x - 6ab + 3yz - 5ax + 11by$.

41. Adding and Subtracting with regard to a Named Letter. $7a + 5a - 6a$ might be written $(7 + 5 - 6)a$. So if we had $7ax + 5bx - 6cx$, we might write it $(7a + 5b - 6c)x$. In this, x has been chosen as the element of likeness or the denomination in the three terms. We have added the coefficients, but in this case the coefficients are unlike and we can only indicate the addition. We can subtract $5ax^2$ from $7bx^2$ if we consider x^2 as the element of likeness. The remainder is $(7b - 5a)x^2$.

EXERCISES.

1. Unite with respect to x, $5\,ax + 7\,xy - 3\,xz$.

2. Unite with respect to a, $3\,ax + 4\,ay - 5\,az$.

3. Unite with respect to k, $4\,xk + 3\,yk - 12\,zk$.

4. Unite with respect to x and y, $4\,ax + 6\,ay - 5\,bx + 12\,cy$.

5. Unite with respect to x and y,

$$3\,ax + 4\,by - (2\,ax + 12\,by).$$

6. Unite with respect to x^2,

$$5\,x^2 + 16\,ax^2 - 12\,bx^2 - x^2.$$

7. Unite with respect to x, y, z,

$$3\,x + 4\,y + 3\,z - (5\,x - 8\,y - 7\,z).$$

$$(3 - 5)\,x + (4 + 8)\,y + (3 + 7)\,z = 2\,x + 12\,y + 10\,z.$$

8. Unite with respect to x and y,

$$3\,x - 2\,y + 7 - (2\,x + 3\,y + 4).$$

9. Unite with respect to x, y, and z,

$$-5\,x + 8\,y + 7\,z + (3\,x - 4\,y + z).$$

10. Unite with respect to the *powers* of x,

$$3\,x^3 - 5\,x^2 - 8\,x^3 + 7\,x^2 - 16\,x + 20\,x^2 - 21\,x + 6.$$

11. Unite with respect to the powers of x and y,

$$ax^2 + by^2 + cx + dy + e - (a'x^2 + b'y^2 + c'x + d'y + e').$$

12. Unite with respect to x^2, xy, and y^2,

$$4\,x^2 - 3\,xy + 12\,y^2 - (2\,x^2 - 4\,xy + 8\,y^2) + 16\,xy.$$

13. Unite with respect to powers of k,

$$12\,x + 16\,yk - 6\,x + 12\,zk + 8\,x^2k^2 + 12\,k^2 + k^3.$$

14. Unite with respect to x^2, y^2, z^2, and xy,

$$ax^2 + 2\,hxy + by^2 + cz^2 - (a'x^2 + 2\,h'xy - b'y^2 + c'z^2).$$

MISCELLANEOUS EXERCISES.

1. A and B have \$550; A has \$100 more than twice as much as B. How much has each?

2. Add $11\,ax^3+13\,bx^2y+9\,cxy^2-5\,dy^3$, $4\,bx^2y-3\,ax^3+16\,dy^3$, $31\,bx^2y-24\,cxy^2$, and $10\,dy^3-8\,cxy^2+ax^3$.

3. Remove the parentheses and simplify,

$$13\,x^2-\{8\,y-5\,x-(3\,x+7\,y)-2\,y\}-(12\,x+\overline{8\,y+2\,x}).$$

4. Unite with respect to x^2 and y^2,

$$ax^2-5\,x^2+3\,y^2-by^2+cx^2-dy^2.$$

5. From $5\,a^3-6\,a^2b-7\,ab^2+11\,b^3$ take $10\,b^3-3\,a^3+5\,ab^2-12\,a^2b$.

6. By means of a diagram show that the sum of 8 and -10 is -2.

7. By means of a diagram show that -10 taken from 8 leaves 18.

8. By means of a diagram show that 8 taken from -10 leaves -18.

9. With $x=4$ and $y=5$, find the value of

$$8\,y-\{3\,x-[4\,y+2\,x-(6\,x-\overline{3\,y-5\,x})]\}.$$

10. With the same values of x and y find the value of

$$\{8\,x-[3\,y+2\,x-(5\,y-3\,x)]\}\,(8\,x-\overline{3\,y-x}).$$

11. A father's age is 10 years more than 3 times his son's age; the sum of their ages is 82 years. Find the age of each.

12. Add, $8\,y+3\,xy+11\,x^2$, $7\,y-13\,xy-21\,x^2$, $-14\,y+10\,xy+12\,x^2$, and $2\,y-5\,xy-2\,x^2$.
Verify your result by putting $y=2$ and $x=1$.

13. From $\quad 11\,a-12\,ab-\ \ 3\,b$
take $\quad \underline{8\,a+\ \ 3\,ab-10\,b+11}$

Verify your result by putting $a=1$ and $b=2$.

14. Add $8(a+b)-16(x-y)+11\sqrt{ax}$, $5(x-y)+2\sqrt{ax}-7(a+b)$, and $-3(a+b)+11(x-y)-14\sqrt{ax}$. (Regard the quantities in each parenthesis as a single term.)

15. From $8(a^2-b^2)-17(x+y)-11\sqrt{x^2+y^2}$
take $\underline{7(a^2-b^2)-20(x+y)+\ \ 4\sqrt{x^2+y^2}}$

16. Show by a diagram that the sum of 8, -6, -3, $+2$, and -5 is -4.

17. Unite with respect to a and x,

$$5a+3x-4a-7x-ba+bx+cx-dx.$$

18. $8x+3y-(x+y)+2yz$. Inclose all the terms after the first in a parenthesis preceded by a $-$ sign.

19. A farmer exchanged a bushels of wheat at b cents a bushel, and c bushels of corn at d cents a bushel, for a mowing machine costing e dollars, and for calves at f dollars each. How many calves did he get?

20. John, James, and Henry together have \$216. John has one half as much as James, and Henry has as much as both John and James. How much has each?

21. Simplify $8x^2+3xy-4y^2+2xy+3y^2-7x^2+4y^2-2xy-7y^2+5x^2+3xy-4y^2+10x^2$.

22. Simplify $17-13x+4x^2-y^3-(3y^3+21x+5x^2-11)$.

23. If $x=1$, $y=2$, and $z=3$, find the value of

$$xyz-\{x^2-[y^2+xz-(2z-\overline{3xy-4y})]+xz^2\}.$$

24. With the same values of x, y, and z, find the value of

$$3yz+\{8x-[4y+z^2-(5xy+4yz)-8xyz]-3y^2\}.$$

25. Add

$$\begin{array}{rrr}
3(x+y)- & 4(a^2-b)+ & 3\sqrt{y^2-b^2}\\
-7(x+y)+ & 11(a^2-b)- & 16\sqrt{y^2-b^2}\\
4(x+y)- & 8(a^2-b)- & 4\sqrt{y^2-b^2}\\
5(x+y)- & 3(a^2-b)- & 9\sqrt{y^2-b^2}\\
\hline
\end{array}$$

CHAPTER IV.

MULTIPLICATION AND DIVISION.

MULTIPLICATION.

42. Integral Multiplication. To multiply 7 by 8 may be taken to mean that 7 is to be added 8 times.

$$7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 = 56.$$

In such an operation 7 is the *multiplicand*, 8 is the *multiplier*, and 56 is the *product*. Similarly, to multiply a by b is to find the sum of a added b times:

$$a + a + a + \cdots (b \text{ times}) = ab.$$

In this, a is the multiplicand, b the multiplier, and ab (read, a multiplied by b) is the product. It is clear that this view of multiplication has no meaning when the multiplier b is fractional or negative. We can not add a one half times, or three fourths times, or three and a half times; neither can we add a negative four times to obtain the product of a by -4. All this shows that if we are to have multiplication of algebraic numbers, we must extend our definition. No change is made in the meaning of multiplicand, multiplier, and product.

43. Multiplication Defined. *Multiplication is doing to the multiplicand what has been done to unity to produce the multiplier.*

For example. To multiply 6 by 5 is to do to 6 what has been done to unity to produce 5. Unity has been added five times $(1+1+1+1+1=5)$ to make 5, hence we must add 6 five times to get the product 6×5. Or, we may say that unity has been taken five times to produce 5, hence we must take 6 five times to produce 6×5. The new definition is seen to agree with the arithmetical notion of multiplying one integer by another. To multiply 6 by $\frac{2}{3}$ is to do to 6 what has been done to unity to produce $\frac{2}{3}$. Unity has been divided into 3 equal parts and 2 of them taken to produce $\frac{2}{3}$. Hence to multiply 6 by $\frac{2}{3}$ we must divide 6 into 3 equal parts and take 2 of the parts. It is thus seen that the new definition includes multiplication of fractions. The application of the definition may be seen by considering the following simple concrete problem.

A water tank with a capacity of 1000 gallons contains at the present time 600 gallons. Let us consider the following four cases:

(1) If it have a supply pipe carrying 10 gallons an hour, how much water will be poured into the tank in the next $8\frac{1}{2}$ hours?

Evidently the amount poured in is $10\times 8\frac{1}{2}$. We must do to 10 what has been done to 1 to make $8\frac{1}{2}$. To produce $8\frac{1}{2}$, 1 has been added eight times, then separated into 2 equal parts and one of the parts added. Hence we must add 10 eight times, giving 80, then separate 10 into two equal parts of 5 each and add the 5 to 80, giving 85. Hence the amount poured in is 85 gallons.

(2) With the same rate of flow, how many gallons must be added to find the amount in the tank 6 hours before the present time?

In this case the 6 hours is negative, as it is the opposite of the time used in the preceding problem, which we considered positive. Our problem now is to multiply 10 by -6. We must do to 10 what

has been done to 1 to produce -6. -6 is produced by subtracting 1 six times. Hence to multiply 10 by -6, we must subtract 10 six times, which gives -60. Hence we must add -60 gallons to 600 gallons to get the contents 6 hours before the present time.

(3) A discharge pipe is flowing at the rate of 5 gallons per hour. How many gallons must be added to give the contents 10 hours from now?

Since the inflow rate was positive, the outflow rate is negative. Hence, our problem now is to multiply -5 by 10. We must do to -5 what has been done to 1 to produce 10; that is we must add -5 ten times, which gives -50. So -50 gallons is the amount to be added.

(4) The discharge pipe has been running at the rate of 6 gallons an hour for a number of hours. How many gallons must be added to give the contents of the tank 8 hours ago?

In this case both the 6 and 8 are negative, and the problem is to multiply -6 by -8. We must do to -6 what has been done to 1 to produce 8; that is, we must subtract -6 eight times, which gives $+48$. So we must add 48 gallons to get the contents of the tank 8 hours ago.

The results of the four cases may be arranged thus:

$$\begin{aligned} 10 \times 8\tfrac{1}{2} &= +85, \\ 10 \times -6 &= -60, \\ -5 \times 10 &= -50, \\ (-6) \times (-8) &= +48. \end{aligned}$$

If we should make our reasoning perfectly general by letting the rate of flow be a and the number of hours be b, then the four cases would take this form:

$$\begin{aligned} a \times b \qquad &= +ab, \\ a \times (-b) \qquad &= -ab, \\ -a \times b \qquad &= -ab, \\ (-a) \times (-b) &= +ab. \end{aligned}$$

44. Law of Signs in Multiplication. From the definition and the above considerations, we see that the product is + if the multiplier and multiplicand have like signs, and it is − if they have unlike signs.

In multiplication, like signs in multiplicand and multiplier give a positive product and unlike signs give a minus product.

45. Continued Products. *Products produced by three or more multiplications are called continued products.*

If 6 boys buy 5 oranges each at 3 cents apiece, the total cost of the oranges is $3 \times 5 \times 6$ cents. If 8 groups of 6 boys each should buy oranges as above, the total cost of the oranges is $3 \times 5 \times 6 \times 8$ cents. If instead of using arithmetical number we should use algebraic number, and say that there were d groups, c boys in a group, and each boy bought b oranges at a cents apiece, then the total cost of the oranges would be $a \times b \times c \times d = abcd$ cents.

46. Factors. As in arithmetic, *the numbers multiplied together are called the factors of the product.*

a, b, and c are factors of abc.

5, a, b, and c are factors of $5\,abc$.

In the latter case, 5 is considered a *numerical multiplier* or *coefficient*, and is usually so designated, instead of being called a factor.

EXERCISES.

Point out the factors and numerical multipliers in the following products:

1. $abcx$.
2. $3\,abdz$.
3. $17 \times 9(3\,z) = 51 \times 9\,z$.
4. $11(2\,x)(3\,y)z$.
5. $(5\,x)(4\,a)(3\,z)$.
6. $(3\,y)(4\,z)(2 \times y)$.

47. Signs of Continued Products.

$$(-3)\times 2\times 3 \qquad\quad = -6\times 3 \qquad = -18.$$
$$(-3)\times(-2)\times 3 \quad\; = +6\times 3 \qquad = +18.$$
$$(-3)\times(-2)\times(-3) = +6\times(-3) = -18.$$

If we use algebraic instead of arithmetical number, the above may be written:

$$(-a)\times b\times c \qquad\quad = (-ab)\times c \qquad = -abc.$$
$$(-a)\times(-b)\times c \quad\; = (+ab)\times c \qquad = +abc.$$
$$(-a)\times(-b)\times(-c) = (+ab)\times(-c) = -abc.$$

We see that one negative factor produces a negative product, two negative factors a positive product, and three negative factors a negative product. Hence,

Products resulting from an even number of negative factors are $+$, *those resulting from an odd number of negative factors are* $-$.

EXERCISES.

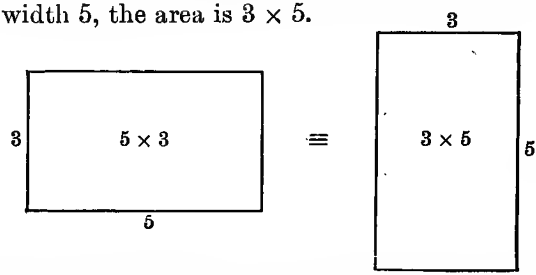
Give the signs of the following products:

1. $(-2a)\times(5a)(-x)$.
2. $(-a)(-b)(-c)(-d)$.
3. $a(-b)(-c)d(-e)f(-g)(-h)$.
4. $x^2(-a)(-b)(y^2)(-3x)$.
5. $17(-5)(-3)(-2)(xyz)(-2)(-3)$.

48. Commutative Law of Factors. In multiplying together numbers in arithmetic, the factors may be taken in any order without changing the product.

$$5\times 7 \equiv 7\times 5 \equiv 35.$$
$$2\times 3\times 5 \equiv 2\times 5\times 3 \equiv 3\times 2\times 5 \equiv 3\times 5\times 2$$
$$\equiv 5\times 2\times 3 \equiv 5\times 3\times 2 \equiv 30.$$

If we construct a rectangle of length 5 and width 3, the area is 5×3. If we construct another rectangle of length 3 and width 5, the area is 3×5.



These two rectangles are readily seen to be equal, for the second one is merely the first one placed on end. Hence,

$$5 \times 3 = 3 \times 5.$$

The reasoning would be exactly similar if we should use the general rectangle whose length is a and width b, and the other general rectangle whose length is b and width a. In this case our conclusion is

$$a \times b = b \times a.$$

The product is the same whatever the order of the factors. This is called the Commutative Law of Factors.

This law holds for all algebraic numbers.

$$a \times (-b) = (-b)a = -(ba) = -(ab) = -ab;$$

or, since $-b = (-1)b$,

$$a \times (-b) = a \times (-1) \times b = (-1)ab = -ab.$$

EXERCISES.

1. Commute the factors of $3xy$.

$$3xy = 3yx = x3y = xy3 = y3x = yx3.$$

2. Commute the factors of $-abc$. (-1 is a numerical multiplier.)

3. Commute the factors of $ab(x+y)$. (The quantity in parenthesis is a factor.)

$ab(x+y)=a(x+y)b=(x+y)ab=(x+y)ba=b(x+y)a=ba(x+y)$.

4. Commute the factors of $(a+b)c(x+y)$.

5. Commute the factors of $(a+b)(c+d)(e+f)$.

49. Associative Law of Factors. If 6 boys buy 4 oranges each at 3 cents apiece, the total cost is 3 cents $\times$ 4 $\times$ 6. We may think of this as each boy paying out 3 cents $\times$ 4 and the 6 boys as paying out (3 cents $\times$ 4) $\times$ 6; or we may think of total number of oranges bought, 4×6, and then of the cost of these as 3 cents $\times$ (4×6); or finally we may think of the six boys as buying 1 orange each, at a cost (3 cents $\times$ 6), and then the total cost of 4 oranges each is (3 cents $\times$ 6) $\times$ 4. Hence, we have

$$3\times 4\times 6=(3\times 4)\times 6=3\times(4\times 6)=(3\times 6)\times 4.$$

If instead of the arithmetical numbers, 3, 4, and 6, we use the algebraic numbers a, b, and c, we have

$$a\times b\times c=(a\times b)\times c=a\times(b\times c)=(a\times c)\times b.$$

The above is an illustration of the *Associative Law of Factors*, which may be stated as follows:

The factors of a product may be grouped in any order.

$$a\times b\times c\times d=(ab)\times(cd)=a(b\times c)d=a(b\times c\times d).$$

$$8\times(-3)\times(-4)\times(-2)=\{[8\times(-3)]\}\times\{(-4)\times(-2)\}=\text{etc.}$$

50. Distributive Law of Factors. The multiplication of $4+5$ by 7 may be indicated thus: $(4+5)\times 7$. The operation may be carried out in either of the following ways:

$$(4+5)\times 7=9\times 7=63.$$

$$(4+5)7=4\times 7+5\times 7=28+35=63.$$

In the first instance the 4 and 5 have been combined into 9, and the product of 9 by 7 taken. In the second instance the 7 has been distributed as a multiplier of the terms 4 and 5 of the multiplicand, and the sum of the separate products taken. If these arithmetical numbers 4, 5, and 7 be replaced by the algebraic numbers a, b, and c, we have

$$(a+b)\times c = a\times c + b\times c = ac + bc.$$

The fact expressed by the algebraic identity

$$(a+b)\times c = ac + bc$$

is known as the *Distributive Law.* Since the multiplier and multiplicand are commutative,

$$c\times(a+b) = (a+b)\times c = ac + bc.$$

It may be noticed that the Distributive Law harmonizes, as it should, with the definition of multiplication. To multiply $a+b$ by c is to do to $a+b$ what has been done to 1 to produce c. This would certainly mean that we are to do the same thing to a and b and add the results.

In distributing factors the law of signs must be observed.

$$(a-b)\times 5c = 5ac - 5bc.$$

$$(-2a^2+b^2)\times(-3c) = 6a^2c - 3b^2c.$$

EXERCISES.

Distribute the following factors:

1. $(a+b-c)\times 2d.$
2. $(2a+3b-4c)\times -(3d).$
3. $(2x+y-3z)(2a).$
4. $(12x^2-5y^2+13xy)(4b).$
5. $(15xy-13yz+12xz)(-2ab).$

6. $(ab - 5b^2 + 3c - a)(5xy)$.

7. $(ax^2 - bx^2 + cxy)(-3m)$.

8. $(4lx + 5my - 3nz)(2ab)$.

9. $(-6a^3 + 5a^2b - 3ab^2 + b^3)(-3x^3)$.

10. $(ax^2 + bx + c)(-2y^2z)$.

51. Index Law of Factors. In a continued product, $abcd$, any two or more of the factors may become equal. To indicate the product when two or more factors have become equal, a convenient notation has been devised. If b should become equal to a in the above product, we would have $aacd$, and it would be written a^2cd. The small ^{2} to the right of the a and slightly elevated is called the *exponent*. If b and c each equal a, we have $aaad = a^3d$. If b, c, and d each equal a, we have $aaaa = a^4$. It should be noticed that the exponents ^{2}, ^{3}, and ^{4} indicate in these cases the number of times a occurs in the respective products.

$$a \cdot a \equiv a^2, \text{ read } a \text{ square.}$$

$$a \cdot a \cdot a \equiv a^3, \text{ read } a \text{ cube.}$$

$$a \cdot a \cdot a \cdot a \equiv a^4, \text{ read } a \text{ fourth power.}$$

$$a \cdot a \cdot a \cdots \text{ to } m\ a\text{'s} \equiv a^m, \text{ read } a \text{ exponent } m, \text{ or } a\ m\text{th.}$$

It should be noted that $a = a^1$.

$$a^5 \equiv a \cdot a \cdot a \cdot a \cdot a.$$

$$a^3x^2 \equiv a \cdot a \cdot a \cdot x \cdot x.$$

Write out the equivalents of

(1) $a^6x^2 \equiv$

(2) $5a^3x^2y^3 \equiv$

(3) $7a \cdot a \cdot a \cdot a \cdot x \cdot x \cdot x \equiv$

(4) $a^7x^2(3)^3 \equiv$

(5) $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \cdot x \cdot x \equiv$

$$a^3 \times a^2 \equiv a \cdot a \cdot a \times a \cdot a = a \cdot a \cdot a \cdot a \cdot a = a^5 = a^{3+2}.$$

$$\begin{aligned} a^m \times a^n &= (a \cdot a \cdot a \cdots \text{ to } m\ a\text{'s})(a \cdot a \cdot a \cdots \text{ to } n\ a\text{'s}) \\ &= (a \cdot a \cdot a \cdots \text{ to } (m+n)\ a\text{'s}) \\ &= a^{m+n}. \end{aligned}$$

The result $a^m \times a^n = a^{m+n}$

is the Index Law for positive integral exponents. It is read a with exponent m multiplied by a with exponent n equals a with exponent $m+n$.

$$x^3 \cdot x^2 = x^{3+2} = x^5.$$

$$y^4 \cdot y^5 = y^{4+5} = y^9.$$

EXERCISES.

1. $x^5 \cdot x^7 \equiv$
2. $a^3 \cdot a^4 \cdot a^5 \equiv (a^3 \cdot a^4) \cdot a^5 \equiv (a^{3+4}) \cdot a^5 = a^{3+4+5} = a^{12}.$
3. $y^7 \cdot y^4 \cdot y^1 \equiv$
4. $a^5 \cdot a^2 \cdot x^4 \cdot x^9 \equiv (a^5 \cdot a^2)(x^4 \cdot x^9) \equiv a^7 \cdot x^{13}.$
5. $3^2 \cdot 3^3 \cdot y^4 \cdot x^5 \cdot y^7 \cdot x^8 \equiv$
6. $2^4 \cdot 2^5 \cdot 3^7 \cdot 3^2 \cdot 5^4 \cdot 5^7 \equiv$
7. $a^{12} \cdot x^5 \cdot y^4 \cdot a^7 \cdot x^{12} \cdot y^9 \equiv$
8. $(x+y)^5 \cdot (x+y)^7 \cdot (x+y)^9 \equiv$
9. $(a+b)^4 \cdot (a+b)^{10} \cdot (x+y)^3 \cdot (x+y)^7 \equiv$
10. $7^3 \cdot 7^6 \cdot (y+z)^3 \cdot (y+z)^{11} \cdot (a-b)^5 \cdot (a-b)^{11} \equiv$

52. The Multiplication of Any Number of Monomials.

RULES:

(1) *Write the product of the numerical coefficients.*

(2) *Attach the literal factors of the product, observing the Index Law for repeated factors.*

(3) *Prefix the proper sign, determined by the number of negative factors, + if an even number, − if an odd number.*

These rules result from observing the laws already developed.

Multiply together $3\,ab$, $4\,ac$, $5\,bc$, $-3\,abc$.

The indicated result is

$3\,ab \times 4\,ac \times 5\,bc \times (-3\,abc)$

$\equiv 3 \cdot 4 \cdot 5 \cdot (-3) a \cdot a \cdot a \cdot b \cdot b \cdot b \cdot c \cdot c \cdot c$ (by Commutative Law)

$\equiv -180\, a \cdot a \cdot a \cdot b \cdot b \cdot b \cdot c \cdot c \cdot c$ (by multiplying numerical factors and observing law of signs)

$\equiv -180\, a^3b^3c^3$ (by Index Law)

The above has been developed by an application of the laws, but the result is in exact accord with the rules on page 48.

$$(2\,a) \times (-3\,ab)(-2\,bc) \equiv +12\,a^2b^2c.$$

EXERCISES.

Multiply together the following monomials:

1. $3\,ax$, $4\,ab$, $-5\,axy$.
2. $5\,a^2x$, $3\,b^2y$, $2\,a^3x^2y^4$.
3. $-4\,m^2n$, $2\,mn^3$, $-5\,m^4n^2x^5$.
4. $-xyz$, $5\,x^2y^3$, $-4\,xy^2z^3$.
5. $-4\,lx^2$, $-3\,l^3y^2$, $-8\,x^4y^5$.
6. $2\,abc$, $3\,b^2c^3$, $-a^4bc^5$.
7. $3(a+b)^3$, $4y(a+b)^2$, $5(a+b)^4$.
8. $8\,ax^2$, $5(a+b)^3$, $-2\,a^3(a+b)^2$.

53. The Multiplication of a Polynomial by a Monomial.

RULES:

(1) *Distribute the monomial as a multiplier of each term of the multiplicand.*

(2) *Connect the results by the proper algebraic signs as determined by the law of signs.*

By the Distributive Law

$$(a+b) \times c = ac + bc.$$

b can be any number, as $d+e$; then $(a+b)c = ac + bc$ becomes

$$(a+d+e)c = ac + (d+e)c = ac + dc + ec.$$

In general,

$$(a+b+c+d+\cdots)m = am + bm + cm + dm + \cdots.$$

EXERCISES.

1. $(5x+3y)\times 2a \equiv 10xa+3ya \equiv 10ax+3ay.$

It is customary to write the letters of a product in alphabetic order. This can always be done by an application of the Commutative Law.

2. $(3a+4b)\times 2a \equiv 6a^2+8ab.$

3. $(3a^2-4ab)\times 4ab \equiv$

4. Multiply $3a^2x-2ax^2+3xy$ by $5axy$.

5. Multiply $-3l^2x+4lx^2-5lxy$ by $-2l^2x^3$.

6. $(ax+3ay-5bcy)\times 3axy \equiv 3a^2xy+9a^2xy^2-15abcxy^2.$

7. Multiply $a^3-a^2b+ab^2-b^3$ by $-a^2b^2$.

8. Multiply $3a^4-4a^3b-6a^2b^2+7ab^3$ by ab^3.

9. Multiply $a^2+b^2+x^2+y^2+z^2$ by $6abx^2$.

10. Multiply $5ab-12cd-3fh$ by $-6bc$.

11. Multiply $-4x^2yz+7xy^2z-3xyz^2$ by $-5xyz$.

12. Multiply $3a^2b^2+7b^2c^2-14c^2a^2$ by $8a^2b^2c^2$.

13. Multiply $3(x+y)^2+4(a+b)^2$ by $2(x+y)(a+b)$.

$$\begin{array}{r} 3(x+y)^2+4(a+b)^2 \\ 2(x+y)(a+b) \\ \hline 6(x+y)^3(a+b)+8(x+y)(a+b^3) \end{array}$$

14. Multiply

$(x^2+y^2)^3-3x(x^2+y^2)^2+4y(x^2+y^2)$ by $5xy(x^2+y^2)$.

15. Multiply $(a-b)^4-3x(a-b)^3+7y(a-b)^2$ by $2x^2y(a-b)^2$.

16. Multiply

$3(x+y)^5+2a^2(x+y)^2-3b^2(x+y)$ by $-a^3b^2(x+y)^4$.

17. Multiply

$5(x^2+y)^3-3a^3(x^2+y)^2-7a^4b(x^2+y)$ by $-4(x^2+y)^2$.

18. Multiply

$$10(a^2+3b^2)^4-6(a^2+3b^2)^2+12(a^2+3b^2) \text{ by } -5(a^2+3b^2)^3.$$

19. Multiply $(a+b+c)^3-3(a+b+c)^2$ by $5(a+b+c)^4$.

20. Multiply

$$(ax^2+bx+c)^3+4(ax^2+bx+c)^2 \text{ by } -5(ax^2+bx+c).$$

54. Meaning of $0 \times a$ and $a \times 0$.

$$(b-c)a = ab - ac.$$

If $b=c$, this becomes

$$(c-c)a = ac - ac.$$

But $\qquad c-c=0$ and $ac-ac=0$.

Hence, $\qquad 0 \times a = 0.$

By the Commutative Law,

$$0 \times a = a \times 0 = 0.$$

55. Multiplication of a Polynomial by a Polynomial. RULE. *To multiply a polynomial by a polynomial multiply each term of the multiplicand by every term of the multiplier and take the algebraic sum of the results.*

This is a direct consequence of the Distributive Law.

Find the product of $(a+b)$ by $(x+y)$.

Let $x+y$ be replaced by c.

Then $(a+b)\times(x+y) \equiv (a+b)\times c \equiv ac+bc.$

But $\qquad c \equiv x+y,$

and so

$$\begin{aligned}(a+b)(x+y) &\equiv ac+bc \equiv a(x+y)+b(x+y)\\ &\equiv ax+ay+bx+by.\end{aligned}$$

This product consists of the sum of the products of each term of the multiplicand by each term of the multiplier.

If three polynomials are to be multiplied together, two of them must be multiplied as above and their product multiplied by the third.

Thus

$$(a+b)(x+y)(r+s) \equiv (ax+ay+bx+by)(r+s)$$

(by multiplying factors one and two together).

$$(ax+ay+bx+by)(r+s) \equiv arx+ary+brx+bry + asx+asy+bsx+bsy.$$

EXERCISES.

Find the product of the following polynomials:

1. $(3a+b)(a+b) \equiv 3a^2+ab+3ab+b^2 \equiv 3a^2+4ab+b^2.$

The distribution may be arranged conveniently as follows:

$$\begin{array}{rl}
3a + b & \\
a + b & \\
\hline
3a^2 + ab & \text{(the product of } (3a+b) \text{ by } a) \\
3ab + b^2 & \text{(the product of } (3a+b) \text{ by } b) \\
\hline
3a^2 + 4ab + b^2 & \text{(the total product)}
\end{array}$$

This is called *long multiplication*, and should only be used when the distribution can not easily be made in the *straight line form* illustrated above.

2. $(x+y)(x-y) \equiv x^2+xy-xy-y^2 \equiv x^2-y^2.$
3. $(2x+y)(2x-y) \equiv 4x^2-y^2.$
4. $(4x+3y)(4x-3y).$
5. $(ax+by)(ax-by).$
6. $(x+4)(x+2).$
7. $(x-2)(x+3).$
8. $(x+5)(x-2).$
9. $(y^2+4)(y^2-3).$
10. $(ax^2+b)(x+c).$
11. $(3x+5)(4x+2).$
12. $(4x^3+3x^2+5x-6)(3x^3-2x^2-3x+2).$

Multiplicand and multiplier are both arranged according to the powers of x; that is, the exponents of x decrease uniformly from left to right. The arrangement will be just as good if we reverse both factors and have the powers of x increase from left to right. For multiplication we may arrange the work as follows:

$$\begin{array}{rrrrrrrl}
4x^3 & +3x^2 & +\ 5x & -\ 6 & & & & \\
3x^3 & -2x^2 & -\ 3x & +\ 2 & & & & \\
\hline
12x^6 & +9x^5 & +15x^4 & -18x^3 & & & & \text{(product of the multiplicand by } 3x^3\text{)} \\
 & -8x^5 & -\ 6x^4 & -10x^3 & +12x^2 & & & (\text{by } -2\,x^2) \\
 & & -12x^4 & -\ 9x^3 & -15x^2 & +18x & & (\text{by } 3\,x) \\
 & & & 8x^3 & +\ 6x^2 & +10x & -12 & (\text{by } 2) \\
\hline
12x^6 & +\ x^5 & -\ 3x^4 & -29x^3 & +\ 3x^2 & +28x & -12 & \text{(the total product)}
\end{array}$$

The orderly arrangement according to powers of x merely insures that the terms of the product will come in an orderly way, thus making it easier to arrange like terms in columns ready for adding. When arranged in this way the highest power in the product appears first, and is the product of the two highest terms in the factors.

13. $(4\,a^2 - 5\,a^3 + 3\,a - 6)(3 - a^2 + 2\,a)$.

When these factors are arranged according to powers of a, we have

$$(-5\,a^3 + 4\,a^2 + 3\,a - 6)(-a^2 + 2\,a + 3).$$

Now multiply as in Exercise 12.

14. $(y^3 - 6\,y + 7\,y^2 - 12)(4 - 2\,y + y^3)$.

When arranged, this exercise becomes

$$(y^3 + 7\,y^2 - 6\,y - 12)(y^3 + 0\,y^2 - 2\,y + 4).$$

Observe that in the multiplier the term y^2 does not occur; in the arrangement according to y we write $0\,y^2$.

15. $(b^4-5+6b^2-3b+2b^3)(b^2-2b+2)$.

16. $(a^3-7a+12a^2+6)(a^2-3a-5)$.

17. $(3x^3+12x^2-10x+4)(-x^2-5+8x)$.

18. $(5x^3y^3-6xy+12x^2y^2-4)(3-4x^2y^2+y^3x^3)$.

19. $(4z^3-4+6z^2-5z)(z^2-4z+3)$.

20. $(3a^2+4a^4+a+5a^3-4)(a^4-a^3+a^2-a+1)$.

21. $(2x^3+3x^2-4x-1)(3x+4)$.

SOLUTIONS.

(1)

$$\begin{array}{rrrrr}
2x^3 + & 3x^2 - & 4x - & 1 & \\
3x + & 4 & & & \\
\hline
6x^4 + & 9x^3 - & 12x^2 - & 3x & \\
 & 8x^3 + & 12x^2 - & 16x - & 4 \\
\hline
6x^4 + & 17x^3 + & 0x^2 - & 19x - & 4
\end{array}$$

(2)

$$\begin{array}{rrrrr}
2 + & 3 - & 4 - & 1 & \\
3 + & 4 & & & \\
\hline
6 + & 9 - & 12 - & 3 & \\
 & 8 + & 12 - & 16 - & 4 \\
\hline
6 + & 17 + & 0 - & 19 - & 4
\end{array}$$

In solution (1) the multiplication is carried out in the usual way, while in solution (2) merely the coefficients are used. In the answer on the right the x's should be inserted, beginning with x^4. The method used on the right is called multiplying by *detached coefficients*. It is a device which saves time by the omission of all letters.

In using detached coefficients the following directions should be observed:

(1) The multiplicand and multiplier must be arranged according to the same letter.

(2) 0 must be used as the coefficient of every power of the letter of arrangement which does not occur.

(3) The letter of arrangement is inserted in the product by beginning at the left with a power equal to the sum of the highest powers in the multiplicand and multiplier, and decreasing uniformly to the right.

The following exercise illustrates this:

22. $(3x^5 + 4x^3 + 5x^2 + 7x + 1) \times (5x^3 - 3x)$.

$$\begin{array}{rrrrrrrr}
3 & +0 & +4 & +5 & +7 & +1 & & \\
5 & +0 & -3 & & & & & \\
\hline
15 & +0 & +20 & +25 & +35 & +5 & & \\
 & & -9 & -0 & -12 & -15 & -21 & -3 \\
\hline
\multicolumn{8}{l}{15x^8 + 0x^7 + 11x^6 + 25x^5 + 23x^4 - 10x^3 - 21x^2 - 3x}
\end{array}$$

The highest powers of x in the two factors are 5 and 3, so the product begins with x^8.

In the following exercises use detached coefficients.

23. $(a^4 - 5a^2 + 4a^3 - 3 + 2a)(3a^2 - 5a + 2)$.

24. $(a^4 - 4a^3 + 5a^2 - 2a + 7)(a^4 + 4a^3 - 5a^2 + 2a - 7)$.

25. $(x^3y^3 - 4x^2y^2 + 6xy - 5)(x^3y^3 + 4x^2y^2 - 6xy + 5)$.

26. $(3x^5 - 7x^3 + 4x - 5)(2x^2 - 3x + 4)$.

27. $(7y^4 - 5y^3 + 3y - 4)(4y^3 - 8y + 1)$.

28. $(a^3b^3 - 6ab + 7)(5ab - 4a^3b^3)$.

29. $(4a^4b^4 - 7 - 6a^2b^2 + 3ab)(ab - 5 + a^3b^3)$.

30. $(3x + 4x - 5)(x^2 + x + 1)(x^2 - 3x + 3)$.

56. The Identity in Multiplication. *The multiplicand multiplied by the multiplier is identically equal to the product.*

$$\text{Multiplicand} \times \text{Multiplier} \equiv \text{Product}.$$

$$(a + b) \times c \equiv ac + bc.$$

Let $$a = b = c = 1.$$

$$(1 + 1) \times 1 = 1 \times 1 + 1 \times 1.$$

$$2 \times 1 = 1 + 1.$$

$$2 = 2.$$

This furnishes a convenient method of verifying the results in multiplication. If in Exercise 22 above we put $x=1$, we have

$$(3+4+5+7+1)\times(5-3)\equiv 15+11+25+23-10-21-3.$$
$$20\times 2\equiv 40.$$
$$40\equiv 40.$$

This merely verifies the coefficients. In order to verify the exponents, some value other than 1 would have to be substituted. In case more than one letter occurs, numerical values must be given to each.

$$(x+y)(x-y)\equiv x^2-y^2.$$

Let $\qquad x=1$ and $y=2$.

Then $\qquad (1+2)(1-2)\equiv 1^2-2^2.$

$$(3)(-1)\equiv 1-4.$$
$$-3\equiv -3.$$

EXERCISES.

Perform the following multiplications and verify by means of the identity:

1. $(a^2+ab)(2a+3b)$.
2. $(4a+6b)(2a-4b)$.
3. $(7x^2+3x+1)(x^2-x+1)$.
4. $(5ab+3cd)(2ab-4cd)$.
5. $(5x^2-3xy+2y^2)(2x+3y)$.
6. $(3x+4a)(3x-4a)$.
7. $(5ab-7cd)(5ab+7cd)$.
8. $(6x^2y-4xy^2+y^3)(3xy-7y^2)$.

57. Involution. In multiplication when the factors are alike, the operation is called *involution*, and the result a *power*.

$$a\cdot a\cdot a=a^3.$$
$$a^2\cdot a^2\cdot a^2=(a^2)^3=aa\;aa\;aa=a^6.$$
$$(a^3\cdot a^3\cdot a^3)=(a^3)^3=aaa\;aaa\;aaa=a^9.$$

58. Meaning of $(a^m)^n$.

$(a^m)^n$ **means** $a^m \cdot a^m \cdot a^m \cdots$ **to** n **factors.**

Since each of the n factors, a^m, contains m a's, there are in the product mn a's.

But a^{mn} means $a \cdot a \cdot a \cdot a \cdots$ to mn factors.

Hence, $$(a^m)^n = a^{mn}.$$

More generally, $$(a^m b^l)^n = a^{mn} b^{ln}.$$

The exponent n is distributive as to *factors* within the parenthesis.

$$(x^3y^4)^3 = x^9y^{12}.$$

The exponent n is *not* distributive as to *terms* within the parenthesis.

$$(x + y)^3 = (x + y)(x + y)(x + y).$$

$$(x + y)^3 \text{ is not equal to } x^3 + y^3.$$

The difference between the following forms should be noted:

$$a^{3^2} = a^9. \qquad a^{m^n} = a^{(m^n)}.$$

$$(a^3)^2 = a^6. \qquad (a^m)^n = a^{mn}.$$

EXERCISES.

Remove the ( ) and simplify:

1. $(a^2)^4(a^3)^2 \equiv a^8 \cdot a^6 \equiv a^{14}.$

2. $(2^2)^2(x^3y)^4.$

3. $((x + y)^2)^4.$

4. $(4^2)^5(x^3y^4)^6.$

5. $2^{3^2} \cdot (2^3)^2 \equiv 2^9 \cdot 2^6 \equiv 2^{15}.$

6. $3^{2^3} \cdot (3^2)^3.$

7. $(a^4x^2)^4(ax^2)^3.$

8. $(3^2)^2 \cdot (2^2)^2(x^3)^2(x^3)^2.$

9. $(x^2 + y^2)^{3^2} \cdot (x^2 + y^2)^{2^3}.$

10. $(ab^2c)^4 \cdot (a^2bc^3)^5.$

DIVISION.

59. Division Defined. — *Division is the process of finding one number, when the product of two numbers and one of them are given. The given product is the dividend, the given number the divisor, and the required number the quotient.*

Division is the inverse of multiplication, the dividend corresponding to the product, the divisor to the multiplier, and the quotient to the multiplicand.

Since $$a \times b = ab,$$
$$ab \div b = a.$$

60. Law of Signs in Division.

From multiplication we have

$$(+a)(+b) = +ab.$$
$$(+a)(-b) = -ab.$$
$$(-a)(+b) = -ab.$$
$$(-a)(-b) = +ab.$$

From the definition of division it follows that

$$+ab \div (+b) = +a.$$
$$-ab \div (-b) = +a.$$
$$-ab \div (+b) = -a.$$
$$+ab \div (-b) = -a.$$

Like signs in dividend and divisor give a positive quotient, and unlike signs give a negative quotient.

61. Index Law. We already know that

$$a^m \times a^n = a^{m+n}.$$

Hence, $$a^{m+n} \div a^n = a^m = a^{m+n-n}.$$

Suppose $$m + n = p,$$

then $$a^p \div a^n = a^{p-n}.$$

More generally, $$a^p b^q c^r \div a^n b^s c^t = a^{p-n} b^{q-s} c^{r-t}.$$

For the present it is understood that the exponents of the factors of the dividend are not less than the exponents of the corresponding factors of the divisor.

The above considerations show that the exponents of the factors of the divisor are subtracted from the exponents of the like factors of the dividend in order to obtain the exponents of the factors of the quotient.

Illustrations:

(1) $a^7 \div a^4 = a^{7-4} = a^3.$

(2) $a^{10}b^6 \div a^3b^5 = a^{10-3}b^{6-5} = a^7b.$

(3) $a^4b^5c^8 \div a^3b^3c^4 = ab^2c^4.$

(4) $(x+y)^3(z+w)^4 \div (x+y)(z+w)^2 = (x+y)^2(z+w)^2.$

62. Meaning of $\frac{0}{a}$ and a^0.

We know that $a \times 0 = 0.$

Hence, $\frac{0}{a} = 0.$

$a^n \times 1 = a^n.$

Hence, $\frac{a^n}{a^n} = 1.$

But, by Index Law, $\frac{a^n}{a^n} = a^{n-n} = a^0.$

Now by Axiom 1, $a^0 = 1.$

Any quantity with an exponent 0 *is equal to* 1.

63. Division of One Monomial by Another Monomial.

RULES:

(1) *Divide the numerical coefficients as in arithmetic.*

(2) *Attach the literal part determined by the Index Law.*

(3) *Prefix the proper sign determined by the law of signs.*

$18\,x^3y^2z^4 \div 6\,xy^2z^3 = 3\,x^2y^0z = 3\,x^2z$, since $y^0 = 1.$

EXERCISES.

Divide:

1. $21\,a^5x^3y^2$ by $7\,a^3x^2y$.
2. $72\,a^4xy^2z^{11}$ by $24\,a^2y^2z^{10}$.
3. $108\,x^7y^4z^2$ by $36\,x^7y^3z^2$.
4. $81\,a^5b^3cd^2$ by $27\,a^5b^2d$.
5. $144\,a^4(x+y)^5$ by $48\,a^3(x+y)^4$.
6. $-63\,(a-b)^5(x-y)^7$ by $21\,(a-b)^4(x-y)^2$.
7. $45\,x^3y^2(x^2+y^2)^2$ by $-5\,y(x^2+y^2)^2$.
8. $3^5\,a^4x^3yz$ by $3^2\,a^2x^3y$.
9. $-7^4\,l^3x^5y^4(z-x)^5$ by $-7^3\,l^2x^5(z-x)^4$.
10. $3^4\cdot 5^3(a-x)^3(b-y)^4$ by $3^2\cdot 5^3(a-x)^2(b-y)^4$.

64. Division of a Polynomial by a Monomial. From the Distributive Law of Factors we know that

$$(a+b+c)\times k = ak+bk+ck.$$

Hence, $\quad (ak+bk+ck)\div k = a+b+c,$

which shows that k is distributed as a divisor to every term of the dividend.

RULE. *Divide each term of the polynomial by the monomial and add the results.*

$$\frac{10\,a^2x^3+5\,ax^2y-20\,a^3x^2y^2}{5\,ax^2} = 2\,ax+y-4\,a^2y^2.$$

EXERCISES.

1. $(12\,a^3y^4-8\,a^4xy^5-4\,a^5x^3y^3)\div 4\,a^3y^3$.
2. $(9\,abcx^4-18\,a^2b^2x^3+27\,a^3bc^2x)\div 9\,abx$.
3. $(30\,x^3y^3z+25\,x^2y^4z^2-35\,x^2y^3z^2)\div 5\,x^2y^2z$.

4. $\dfrac{(x+y)^3+5(x+y)^2-a(x+y)}{(x+y)}$. (Regard $x+y$ as a term.)

5. $[4(a-b)^4-5x(a-b)^3+11xy(a-b)^5]\div(a-b)^3$.

6. Divide $5xy(x+y)^4-10x^3y^2(x+y)^3-15xy^2(x+y)^5$ by $5xy(x+y)^2$.

7. Divide $11a^2b^2(x^2+a^2)+22a^3b(x^2+a^2)^5-33ab^3(x^2+a^2)^2$ by $11ab(x^2+a^2)$.

8. Divide

$$7x^3yz(a^2-b^2)^2+21xy^2z^2(a^2-b^2)^{11} \text{ by } -7xyz(a^2-b^2)^2.$$

9. Divide

$$24a^3b^2c(ax+b)^4-36a^2b^4c^5(ax+b)^7 \text{ by } -12(a^2b^2c)(ax+b)^4.$$

10. Divide

$$-33(ax^2+bx+c)^2+44a^4(ax^2+bx+c)^5 \text{ by } 11(ax^2+bx+c)^2.$$

65. Division of a Polynomial by a Polynomial. The dividend and divisor should be arranged in descending or ascending powers of some common leading letter. This gives a quotient arranged with respect to the same letter.

The first term of the quotient is found by dividing the first term of the dividend by the first term of the divisor. The process is illustrated in the following solutions:

(1) Divide

$$x^5-x^4-11x^3+16x^2-2x-3 \text{ by } x^2-4x+3.$$

$$\begin{array}{rll}
x^2-4x+3)\,x^5-x^4-11x^3+16x^2-2x-3 & (x^3+3x^2-2x-1 & \\
\underline{x^5-4x^4+3x^3} \qquad\qquad\qquad & =x^3(x^2-4x+3) & \\
3x^4-14x^3+16x^2-2x-3 & =\text{1st partial div.} & \\
\underline{3x^4-12x^3+9x^2} \qquad\qquad & =3x^2(x^2-4x+3) & \\
-2x^3+7x^2-2x-3 & =\text{2d partial div.} & \\
\underline{-2x^3+8x^2-6x} \quad & =-2x(x^2-4x+3) & \\
-x^2+4x-3 & =\text{3d partial div.} & \\
\underline{-x^2+4x-3} & =-1(x^2-4x+3) &
\end{array}$$

This scheme of division is merely a separation of the dividend into parts. In the example just solved we have separated $x^5 - x^4 - 11x^3 + 16x^2 - 2x - 3$ into these parts:

$$(x^5 - 4x^4 + 3x^3) + (3x^4 - 12x^3 + 9x^2) + (-2x^3 + 8x^2 - 6x) + (-x^2 + 4x - 3).$$

Now, regarding the dividend in this separated form, we have the division thus:

$$\frac{x^5 - 4x^4 + 3x^3}{x^2 - 4x + 3} + \frac{3x^4 - 12x^3 + 9x^2}{x^2 - 4x + 3} + \frac{-2x^3 + 8x^2 - 6x}{x^2 - 4x + 3} + \frac{-x^2 + 4x - 3}{x^2 - 4x + 3} = x^3 + 3x^2 - 2x - 1.$$

(2) Divide $x^4 - 16$ by $2 + x$.

$$\begin{array}{r|l}
x + 2)\,x^4 - 16 & (x^3 - 2x^2 + 4x - 8 \\
\underline{x^4 + 2x^3} & \\
-2x^3 & \\
\underline{-2x^3 - 4x^2} & \\
+4x^2 & \\
\underline{+4x^2 + 8x} & \\
-8x - 16 & \\
\underline{-8x - 16} &
\end{array}$$

Divisions such as the above, which terminate without any remainder, are called *exact*.

EXERCISES.

Divide:

1. $x^4 - x^3 - 9x^2 + 13x - 12$ by $x - 3$.
2. $x^4 - 5x^3 - 3x + 15$ by $x - 5$.
3. $6x^3 + 7x^2 - 18x + 5$ by $2x + 5$.
4. $2x^4 - 9x^3 + 17x^2 - 14x$ by $x^2 - 2x$.
5. $3a^4 - 6a^3 + 2a^2 + 14a - 21$ by $a^2 - 2a + 3$.

6. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ by $a^2 + 2ab + b^2$.
7. $x^6 - a^6$ by $x^2 - a^2$.
8. $b^2c^6 - a^8$ by $bc^3 + a^4$.
9. $m^6 - 3m^4 + 3m^2 - 1$ by $m^2 - 1$.
10. $a^{2n} + 2a^nb^n + b^{2n}$ by $a^n + b^n$.
11. $a^{2n} - b^{4n}$ by $a^n - b^{2n}$.
12. $a^{3n} - b^{3n}$ by $a^n - b^n$.
13. $x^4 - 13x^3 + 47x^2 - 31x + 4$ by $x^2 - 6x + 1$.
14. $x^4 - 12x^3 + 54x^2 - 108x + 81$ by $x^2 - 6x + 9$.
15. $a^3b^3 - 3a^2b^2cd + 3abc^2d^2 - c^3d^3$ by $ab - cd$.
16. $12x^6y^6 - 17x^4y^4 + 10x^2y^2 - 3$ by $4x^2y^2 - 3$.
17. $x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$ by $x^2 - 4x + 4$.
18. $7 + 15x - 21x^2 + 18x^3 - 4x^4$ by $7 - 6x + 4x^2$.
19. $(a+b)^2 - 5(a+b) + 4$ by $(a+b) - 1$.
20. $(x+y)^2 + 7(x+y) - 18$ by $(x+y) + 9$.
21. $(a+x)^2 - (a+x) - 42$ by $(a+x) - 7$.
22. $(m+n)^2 - 7(m+n) - 44$ by $(m+n) + 4$.
23. $(a+b)^3 - x^3$ by $(a+b) - x$.
24. $(x+y)^3 - (a+b)^3$ by $(x+y) - (a+b)$.

66. Detached Coefficients. When the dividend and divisor are arranged in descending powers of some common letter, the quotient is also thus arranged. We may then perform the division by the use of the coefficients only.

(1) Divide $x^4 + x^3 - 3x^2 + 7x - 6$ by $x^2 - x + 2$.

$$\begin{array}{r}
1-1+2)1+1-3+7-6(1+2-3 \\
\underline{1-1+2} \\
2-5+7 \\
\underline{2-2+4} \\
-3+3-6 \\
\underline{-3+3-6}
\end{array}$$

Since $x^4 \div x^2 = x^2$, we know that the quotient begins with x^2, and is $x^2 + 2x - 3$.

In the use of detached coefficients all powers of the letters from the highest to the lowest power must be present in both divisor and dividend. If any powers are absent, they must be inserted with zero coefficients.

(2) Divide $x^3 - 8$ by $x - 2$.

In the dividend neither x^2 nor x appears. We insert them, writing the dividend $x^3 + 0x^2 + 0x - 8$.

$$\begin{array}{l}
1-2)1+0+0-8(1+2+4 \\
\qquad\quad \underline{1-2} \qquad\qquad x^2+2x+4, \text{ quotient.} \\
\qquad\qquad\quad 2+0 \\
\qquad\qquad\quad \underline{2-4} \\
\qquad\qquad\qquad\quad 4-8 \\
\qquad\qquad\qquad\quad \underline{4-8}
\end{array}$$

EXERCISES.

Divide, solving by detached coefficients:

1. $x^4 - 5x^2 + 4$ by $x - 1$; by $x + 2$.
2. $x^4 - 7x^3 + 11x^2 + 7x - 12$ by $x - 1$; by $x - 4$.
3. $x^4 - 13x^2 + 36$ by $x^2 + x - 6$.
4. $x^4 - 18x^2y^2 - 175y^4$ by $x^2 - 25y^2$.
5. $2m^4 - 17m^3n + 31m^2n^2 - 23mn^3 + 12n^4$ by $2m - 3n$.
6. $a^4 - 256$ by $a^2 + 16$.
7. $b^6 - 729$ by $b^2 - 9$.
8. $y^8 - 4096$ by $y^2 + 8$.
9. $a^8 + a^4 + 1$ by $a^4 - a^2 + 1$.
10. $16a^4 + 4a^2 + 1$ by $4a^2 + 2a + 1$.
11. $x^4 + 4$ by $x^2 + 2x + 2$.
12. $13b + 15b^3 - 17b^2 - 3$ by $5b^2 - 4b + 3$.

13. $a^6 - b^6$ by $a^3 + 2a^2b + 2ab^2 + b^3$.

14. $x^5 + y^5$ by $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

15. $10a^4 - 48a^3b + 26a^2b^2 + 24ab^3$ by $-5a^2 + 4ab + 3b^2$.

67. Inexact and Continued Division.

(1) Divide $x^2 + 1$ by $x + 1$.

$$
\begin{array}{r|l}
x+1)\,x^2+1 & (x-1 \\
x^2+x & \\
\hline
-x+1 & \\
-x-1 & \\
\hline
2 &
\end{array}
$$

In this example there is a remainder of 2, and the division is inexact. In such examples the division should continue until the largest exponent of the remainder is less than the largest exponent of the divisor.

(2) Divide $1 + x^2$ by $1 + x$.

$$
\begin{array}{r|l}
1+x)\,1+x^2 & (1-x+2x-2x^3 \\
1+x & \\
\hline
-x+x^2 & \\
-x-x^2 & \\
\hline
2x^2 & \\
2x^2+2x^3 & \\
\hline
-2x^3 & \\
-2x^3-2x^4 & \\
\hline
2x^4 &
\end{array}
$$

This division may end with two terms of the quotient and the remainder $2x^2$, or with three terms and the remainder $-2x^3$, or with four terms and the remainder $2x^4$. Evidently the division might be continued to any number of terms desired. When inexact division takes this form, it is called *continued division.*

EXERCISES.

Find quotients and remainders:

1. Divide $x^3 - 7x^2 + 11x - 7$ by $x + 3$.
2. Divide $x^4 + 7x^3 - 8x + 13$ by $x^2 + 3x - 2$.
3. Divide $a^3 - 11a^2 + 21$ by $a + 5$.
4. Divide $b^4 - 14b^2 + 11$ by $b^2 + 9$.
5. Divide $a^3x^3 + 9a^2x^2 - 7$ by $ax + 3$.

In the next five exercises continue the division to four terms:

6. Divide $1 + x$ by $1 - x$.
7. Divide $2 + 3x - 4x^2$ by $2 - x$.
8. Divide $3 - 6x + 8x^2$ by $3 + 5x$.
9. Divide $1 - 17x + 13x^2 - 8x^3$ by $1 - 16x + 5x^2$.
10. Divide $10 - 20x + 25x^2 - 31x^3$ by $2 - 5x$.

68. The Identity in Division.

$$\text{Dividend} \div \text{Divisor} \equiv \text{Quotient}.$$

$$(x^3 - 1) \div (x - 1) \equiv x^2 + x + 1.$$

This is true for all values of x.

Let $x = 2$, and we have

$$\begin{aligned}
(2^3 - 1) \div (2 - 1) &\equiv 2^2 + 2 + 1,\\
(8 - 1) \div 1 &\equiv 4 + 2 + 1,\\
7 \div 1 &\equiv 7,\\
7 &\equiv 7.
\end{aligned}$$

In this case, should we make $x = 1$, we get $0 \div 0$ on the left of the sign $\equiv$. $0 \div 0$ is indeterminate. In using the identity to verify divisions, avoid substitutions that will produce this form.

EXERCISES.

Divide and verify by substituting particular values:

1. $6l^3 - 17l^2 + 24l - 16$ by $2l^2 - 3l + 4$.

2. $y^5 - 1$ by $y - 1$.

3. $x^5 + 1$ by $x + 1$.

4. $a^3 - 12a^2 + 48a - 64$ by $a^2 - 8a + 16$.

5. $625 - 500z + 150z^2 - 20z^3 + z^4$ by $5 - z$.

6. $625 - z^4$ by $25 + z^2$.

7. $y^4 - 3y^2 - 154$ by $y^2 - 14$.

8. $a^4b^4 + 4a^2b^2 - 117$ by $a^2b^2 + 13$.

9. $x^6 - y^6$ by $x^2 - y^2$.

10. $a^4 - 4a^3 - 34a^2 + 76a + 105$ by $a - 7$.

REVIEW EXERCISES.

1. Find the value of $6x^2 - 4xy + 12y^2$, when $x = 4$, $y = -1$.

2. Find the value of $x^3 - 64y^3 + 8z^3 - 3xyz$, when $x = 0$, $y = 2$, $z = 5$.

3. From $16x^3 - 4y^3 + 12z^3 - 14x^2y + 3xy^2$ subtract $12y^3 + 8x^3 + 4xy^2 - 10x^2y + 11z^3$.

4. Remove parentheses and unite like terms:

$$16 - \{12a + [4b - 3c] + 8 - [8a - 3(4 - 2b)]\}.$$

5. Remove parentheses and unite like terms:

$$-3x^2 + 4[xy - x(3x - 4y) - 3y(4x + 2y)] - \{x^2 + 3(xy - y^2)\}.$$

6. Unite terms in x, y, z:

$$ax + by + cz - 4(-a'x + b'y - c') + 3(lx + my + nz).$$

7. Simplify $x - 4y - [z - y - (x + y - z)]$, and find value when $x = y = z = 1$.

8. Simplify $4(a-5\{b-c\})-[3b+\{2b-(c-a)\}]$, and find value when $a=1$, $b=2$, $c=3$.

9. Find value of $\sqrt{x^2+yz}+\sqrt{y^2+2zx}+\sqrt{z^2+xy}-\sqrt{xyz}$, when $x=1$, $y=0$, $z=2$.

10. Find value of $\dfrac{3\sqrt{x^2+y^2}-4\sqrt{x^2+y^2+z^2-xy-yx-zx}}{x+y-z}$, when $x=4$, $y=5$, $z=6$.

11. Remove parentheses and simplify:

$$x(y-z)+y(z-x)+z(x-y).$$

12. Remove parentheses and simplify:

$$x^2(y-z)+y^2(z-x)+x^2(z-y).$$

13. Multiply x^2+x+1 by x^2-x+1.

14. Multiply $x^2+y^2+1-x-y-xy$ by $x+y+1$.

15. Multiply x^2-4y^2 by x^2+xy^2.

16. Multiply $12x^4-3x^3+10x^2-5x+4$ by $3x^3-x^2+5x-4$.

17. Find the value of x^3-4x^2+3x-5, (1) when $x=2$; (2) when $x=-1$; (3) when $x=0$.

18. Find the remainder after dividing x^3-4x^2+3x-5 (1) by $x-2$; (2) by $x+1$; (3) by x.

Note that the remainders are the same as the results found in Exercise 17.

19. Divide $2a^5-3a^4b-6a^3b+13a^2b^2-6ab^3$ by $2a-3b$.

20. Divide $3x^4+14x^3+9x+2$ by x^2+5x+1.

21. Divide $2a^2+ab-ac-3b^2-4bc-c^2$ by $2a+3b+c$.

22. Divide $x^2-(a+b)x+ab$ by $x-a$.

23. Divide $x^3-(a+b+c)x^2+(ab+bc+ca)x-abc$ by $x-a$.

24. Divide $z^3-(l+m+n)z^2+(lm+mn+ln)z-lmn$ by $z^2-(l+n)z+ln$.

25. Multiply together $(x^n + a^n)$ and $(x + a)$.

26. Multiply $x^n - y^n$ by $x^n + y^n$.

27. Multiply $3x^{2n} + 5x^n + 7$ by $2x^n - 4x - 3$.

28. Divide $3x^{2n} + 13x^{2n-1} + 15x^{2n-2} + 9x^{2n-3}$ by $x^n + 3x^{n-1}$.

29. Divide $x^{m+n} + y^n x^m - y^m x^n - y^{m+n} + x^n + y^n$ by $x^n + y^n$.

30. Multiply $a^3 + b^3$ by $a - b$, and divide the product by $a + b$.

31. Multiply $3x^p + 4x^q - 2x^r$ by $2x^p - 3x^q + x^r$.

32. Divide $a^{3n} - b^{3n}$ by $a^n - b^n$.

33. Multiply $x^{n+2} + 3x^{n+1} - 5x^n$ by $x^{n-2} - 2x^{n-1} + x^n$.

34. Divide $x^{5n} - y^{5n}$ by $x^{4n} + x^{3n}y^n + x^{2n}y^{2n} + x^n y^{3n} + y^{4n}$.

35. Divide $a^{6n} - b^{12n}$ by $a^{2n} - b^{4n}$.

36. Divide $(a + b)^{2n} - x^{4r}$ by $(a + b)^n + x^{2r}$.

37. Divide $(x^2 + y)^{3n} - 1$ by $(x^2 + y)^n - 1$.

38. $(\sqrt{ax + b})^3 + y^3$ by $\sqrt{ax + b} + y$.

39. Simplify $(x^2 - xy + y^2)(x + y)(x - y)^2 \div (x^3 + y^3)$.

40. $5(a^3 + a^2b + ab^2 + b^3) \times 4(a - b)^2 \div [2(a^2 - b^2) \times 10(a^2 + b^2)]$.

CHAPTER V.

IMPORTANT IDENTITIES.

Many expressions in algebra appear in *standard* or *type* forms. When this is the case, multiplications and divisions can be performed mentally by remembering certain identities.

69. Multiplication Identities.

1. *The Product of Two Binomials with a Common Term.*

This form is given by

$$(x+a)(x+b) \equiv x^2 + (a+b)x + ab,$$

or $$(a+x)(b+x) \equiv ab + (a+b)x + x^2.$$

(1) $$(x+10)(x+5) \equiv x^2 + (10+5)x + 50$$
$$\equiv x^2 + 15x + 50.$$

To find the product of $x+10$ and $x+5$ it is only necessary to see that a is 10 and b is 5 and make these substitutions.

(2) $$(x-7)(x+8) \equiv x^2 + (-7+8)x - 56$$
$$\equiv x^2 + x - 56.$$

In this a is -7, and b is 8.

(3) $$(5+x)(11-x) \equiv 55 + (11-5)x - x^2.$$

In this exercise it is necessary to note the signs of the x's.

(4) $$(3x+y-5)(3x+y+7) \equiv (3x+y)^2 + 2(3x+y) - 35.$$

In this exercise the common part is $3x+y$.

EXAMPLES.

Write out the products of the following:

1. $(x+10)(x-2)$.
2. $(3x+6)(3x+1)$.
3. $(7a-5)(7a+4)$.
4. $(5+2a)(5+6a)$.
5. $(2y-6)(2y+7)$.
6. $(ax-11)(ax+6)$.
7. $(3xy-5)(3xy-6)$.
8. $(4x^2+7)(4x^2-5)$.
9. $(3-4xy)(3+6xy)$.
10. $(a+b-6)(a+b+5)$.
11. $(a+b+7)(a+b-8)$.
12. $(x+2ab-3)(x+2ab+7)$.
13. $\{4-(x+2y)\}\{5+(x+2y)\}$.
14. $[(a+b)^2-4x][(a+b)^2+7x]$.
15. $[3ab-(x-y)^2][5ab+(x-y)^2]$.

2. *The Square of a Binomial Sum.*

If, in the identity

$$(x+a)(x+b)\equiv x^2+(a+b)x+ab,$$

we let $$b=a,$$

it becomes $$(x+a)(x+a)\equiv x^2+(a+a)x+aa.$$

$$\boldsymbol{(x+a)^2\equiv x^2+2ax+a^2.}$$

The square of the sum of two quantities is equal to the sum of their squares increased by twice their product.

$$(5x+4y)^2\equiv(5x)^2+2(5x)(4y)+(4y)^2$$
$$\equiv 25x^2+40xy+16y^2.$$

EXERCISES.

Write out the results in the following:

1. $(x+y)^2$.
2. $(2x+a)^2$.
3. $(3x+4b)^2$.
4. $[(x+y)+a]^2\equiv(x+y)^2+2a(x+y)+a^2$.
5. $[y+(a+b)]^2$.
6. $[3x+(2a+c)]^2$.
7. $[(a-3)+5y]^2$.
8. $[(a+b)+(x+y)]^2$.

3. *The Square of a Binomial Difference.*

If, in the identity

$$(x + a)^2 \equiv x^2 + 2\,ax + a^2,$$

we change a to $-a$, it becomes

$$(x - a)^2 \equiv x^2 - 2\,ax + a^2.$$

The square of the difference of two quantities is equal to the sum of their squares diminished by twice their product.

$$(2\,a - y)^2 \equiv (2\,a)^2 - 2\,(2\,a)\,y + y^2$$
$$\equiv 4\,a^2 - 4\,ay + y^2.$$

EXERCISES.

Write out the results in the following:

1. $(a - x)^2$.
2. $(3\,a - y)^2$.
3. $(xy - 4\,b)^2$.
4. $(3\,a^2 - by)^2$.
5. $[(a + b) - xy]^2$.
6. $[(3\,x + y) - ab]^2$.
7. $[(x + y) - (a + b)]^2$.
8. $[(2\,x - y) - 3\,x^2y]^2$.

4. *The Product of the Sum and Difference of Two Quantities.*

If, in the identity

$$(x + a)(x + b) \equiv x^2 + (a + b)\,x + ab,$$

we let $b = -a$,

it becomes $(x + a)(x - a) \equiv x^2 + (a - a)\,x - aa.$

$$(x + a)\,(x - a) \equiv x^2 - a^2.$$

The product of the sum and difference of two quantities is equal to the difference of their squares.

(1) $(7\,x - 3\,y)(7\,x + 3\,y) = (7\,x)^2 - (3\,y)^2 = 49\,x^2 - 9\,y^2.$

(2) $(ax + by - c)(ax + by + c) = (ax + by)^2 - c^2.$

EXERCISES.

Write out the results in the following:

1. $(x-y)(x+y)$.
2. $(3a-b)(3a+b)$.
3. $(4x+ab)(4x-ab)$.
4. $[(a+x)+a][(a+x)-a] \equiv (a+x)^2 - a^2 = a^2 + 2ax + x^2 - a^2$
$\equiv 2ax + x^2$.
5. $[(x+2y)-x][(x+2y)+x]$.
6. $(x^2+2x+16)(x^2+2x-16)$.
7. $[(a+b)-(x+y)][(a+b)+(x+y)]$.
8. $(ax^2+bx+c)(ax^2+bx-c)$.

5. *The Square of a Polynomial.*

If, in the identity

$$(x+a)^2 \equiv x^2 + 2ax + a^2$$

we put $a = y + z$,

it becomes $(x+y+z)^2 \equiv x^2 + 2x(y+z) + (y+z)^2$.

$$\boldsymbol{(x+y+z)^2 \equiv x^2+y^2+z^2+2xy+2xz+2yz.}$$

This may easily be extended to include the square of a polynomial of any number of terms, the result being that

The square of a polynomial equals the sum of the squares of each term increased by twice the product of each term by every other term.

(1) $(x+y-a)^2 \equiv x^2+y^2+a^2+2xy+2x(-a)+2y(-a)$
$\equiv x^2+y^2+a^2+2xy-2ax-2ay$.

(2) $(3x+2a-4b)^2$
$\equiv (3x)^2+(2a)^2+(-4b)^2+2(3x)(2a)+2(3x)(-4b)$
$+2(2a)(-4b)$
$\equiv 9x^2+4a^2+16b^2+12ax-24bx-16ab$.

EXERCISES.

Write out the results in the following:

1. $(a - x - y)^2$.
2. $(3x - y + b)^2$.
3. $(2a + 3b - y)^2$.
4. $(3a - 5x - 2y)^2$.
5. $(2 + 3a - 4b)^2$.
6. $(x - y + a + b)^2$.
7. $(m - n - p - q)^2$.
8. $(2x + y - 3z + a)^2$.
9. $(a^2 - ab + 2x - 3y)^2$.
10. $(2x - y^2 + xy - a)^2$.

6. *The Product of Three Binomials.*

By actual multiplication, it is found that

$$(\boldsymbol{x} + \boldsymbol{a})(\boldsymbol{x} + \boldsymbol{b})(\boldsymbol{x} + \boldsymbol{c}) \equiv \boldsymbol{x}^3 + (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})\boldsymbol{x}^2$$
$$+ (\boldsymbol{ab} + \boldsymbol{bc} + \boldsymbol{ca})\boldsymbol{x} + \boldsymbol{abc}.$$

The product is arranged according to the powers of the common letter x. The coefficient of x^2 is the *algebraic* sum of the second terms of the binomials, the coefficient of x is the *algebraic* sum of their products in pairs, and the term free from x is the product of the second terms of the binomials.

(1) $(x + 1)(x + 2)(x + 3)$
$$\equiv x^3 + (1 + 2 + 3)x^2 + (1 \times 2 + 1 \times 3 + 2 \times 3)x + 1 \times 2 \times 3$$
$$\equiv x^3 + 6x^2 + 11x + 6.$$

(2) $(x + 2)(x - 3)(x + 4)$
$$\equiv x^3 + (2 - 3 + 4)x^2 + [2(-3) + 2 \times 4 + (-3)(4)]x$$
$$+ 2(-3)(4)$$
$$\equiv x^3 + 3x^2 - 10x - 24.$$

EXERCISES.

Write out the results in the following:

1. $(x + c)(x + d)(x + e)$.
2. $(a + x)(a + y)(a + z)$.
3. $(b + 2)(b - 1)(b + 3)$.
4. $(y - 3)(y + 2)(y - 1)$.

5. $(m+5)(m-4)(m-3)$.

6. $(xy+2)(xy-7)(xy-1)$.

7. $(x^2-5)(x^2-3)(x^2+8)$.

8. $(y^3-4)(y^3+11)(y^3-7)$.

9. $(x+y+5)(x+y+3)(x+y+2)$.

10. $(3x+y-2)(3x+y-4)(3x+y+6)$.

11. $(ax+b+2)(ax+b+8)(ax+b-5)$.

12. $(ax^2+bx+4)(ax^2+bx-2)(ax^2+bx-1)$.

7. *The Cube of a Binomial.*

If, in the identity

$$(x+a)(x+b)(x+c) \equiv x^3+(a+b+c)x^2 + (ab+bc+ca)x+abc,$$

we put b and c each equal to a, it becomes

$$(x+a)(x+a)(x+a) \equiv x^3+(a+a+a)x^2 + (aa+aa+aa)x+aaa,$$

or

$$\begin{aligned}(\boldsymbol{x}+\boldsymbol{a})^3 &\equiv \boldsymbol{x}^3+\mathbf{3}\,\boldsymbol{a}\boldsymbol{x}^2+\mathbf{3}\,\boldsymbol{a}^2\boldsymbol{x}+\boldsymbol{a}^3\\ &\equiv \boldsymbol{x}^3+\boldsymbol{a}^3+\mathbf{3}\,\boldsymbol{a}\boldsymbol{x}(\boldsymbol{x}+\boldsymbol{a}).\end{aligned}$$

If, in the above identity, we change a to $-a$, we have

$$\begin{aligned}(\boldsymbol{x}-\boldsymbol{a})^3 &\equiv \boldsymbol{x}^3-\mathbf{3}\,\boldsymbol{a}\boldsymbol{x}^2+\mathbf{3}\,\boldsymbol{a}^2\boldsymbol{x}-\boldsymbol{a}^3\\ &\equiv \boldsymbol{x}^3-\boldsymbol{a}^3-\mathbf{3}\,\boldsymbol{a}\boldsymbol{x}(\boldsymbol{x}-\boldsymbol{a}).\end{aligned}$$

The cube of a binomial is the cube of the first term plus three times the algebraic product of the square of the first term and the second term, plus three times the algebraic product of the first term and the square of the second term, plus the algebraic cube of the second term.

(1) $$\begin{aligned}(x+2y)^3 &\equiv x^3+3(x)^2(2y)+3(x)(2y)^2+(2y)^3\\ &\equiv x^3+6x^2y+12xy^2+8y^3.\end{aligned}$$

(2) $$\begin{aligned}(2a-3b) &\equiv (2a)^3+3(2a)^2(-3b)+3(2a)(-3b)^2 + (-3b)^3\\ &\equiv 8a^3-36a^2b+54ab^2-27b^3.\end{aligned}$$

EXERCISES.

Write out the results in the following:

1. $(2a+b)^3$.
2. $(a-3b)^3$.
3. $(3x+4)^3$.
4. $(2x-5y)^3$.
5. $(3x+ab)^3$.
6. $(2mn-pq)^3$.
7. $(5x^2-1)^3$.
8. $(6y-\frac{1}{6})^3$.
9. $(4ab-5y)^3$.
10. $\left(\frac{x}{2}+\frac{2}{x}\right)^3$.
11. $[(x+y)-2]^3$.
12. $[(x-y)+a]^3$.
13. $[2(x+y)-3a]^3$.
14. $[(ax+b)+c]^3$.
15. $[(a+b)+(x+y)]^3$.
16. $[(ax+b)-(cx+d)]^3$.

8. *The Binomial Theorem.*

$$(a+b)^n \equiv a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3$$
$$+ \cdots\cdots + nab^{n-1} + b^n.$$

The above identity is known as the *Binomial Theorem.* A general proof for it will not be given. For the present we will limit the exponent n to integral values.

If we multiply both sides of the identity

$$(a+b)^3 \equiv a^3 + 3a^2b + 3ab^2 + b^3 \text{ by } a+b,$$

we have $(a+b)^4 \equiv a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

If in the *binomial theorem* we make $n = 4$, we have

$$(a+b)^4 \equiv a^4 + 4a^3b + \frac{4\cdot 3}{1\cdot 2}a^2b^2 + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}ab^3 + \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}b^4$$
$$\equiv a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,$$

a result which agrees with that found by multiplying.

If we multiply both sides of the identity

$$(a+b)^4 \equiv a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4 \text{ by } a+b,$$

we have

$$(a+b)^5 \equiv a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5.$$

If, in the *binomial theorem*, we make $n = 5$, we have

$$(a+b)^5 \equiv a^5 + 5\,a^4b + \frac{5\cdot 4}{1\cdot 2}a^3b^2 + \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}a^2b^3 + \frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}ab^4 + \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5}b^5$$

$$\equiv a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5,$$

a result which agrees with that found by multiplying.

The identity

$$(a+b)^n \equiv a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots\cdots + nab^{n-1} + b^n$$

is often called the *binomial expansion.*

The following *Laws* should be observed in regard to the *Exponents* and *Coefficients* of the successive terms of the binomial expansion.

(1) *Law of Exponents.* The sum of the exponents of a and b in any term is always n; the *leading* letter a appears in the first term with the exponent n which decreases by unity in each succeeding term; the letter b appears in the second term with the exponent 1 which increases by unity in each succeeding term to the last term b^n.

(2) *Law of Coefficients.* If any term be taken, the *coefficient* of the next succeeding term is obtained by multiplying the coefficient of the given term by the exponent of the leading letter a, and dividing this product by the number of the given term in the series.

Thus, in

$$(a+b)^4 \equiv a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4,$$

the coefficient 6 in the third term is obtained by taking the product of 4, the coefficient of the preceding term, by the exponent of a, 3, giving 4×3, and dividing by 2, the number of the term $4\,a^3b$ in the series. Hence, the coefficient of the third term is $4 \times 3 \div 2 = 6$.

Pascal's Triangle. The coefficients of the terms in the expansion of

$$(a+b)^1,\ (a+b)^2,\ (a+b)^3,\ (a+b)^4, \text{ etc.},$$

may be arranged in a table forming what has been called Pascal's Triangle. The arrangement follows.

Coefficients of $(a+b)^1$ are	1	1				
Coefficients of $(a+b)^2$ are	1	2	1			
Coefficients of $(a+b)^3$ are	1	3	3	1		
Coefficients of $(a+b)^4$ are	1	4	6	4	1	
Coefficients of $(a+b)^5$ are	1	5	10	10	5	1

etc.

Each number appears as the sum of the number immediately above and the one to the left. Thus, the first 10 is the sum of 6 and 4; the second 10 is the sum of 4 and 6; the last 5 is the sum of 1 and 4. By this simple arrangement the binomial coefficients for any power of $a+b$ may be easily written out, provided we know the coefficients of the expansion of $a+b$ for a power *one* lower. Knowing

1 5 10 10 5 1

to be the coefficients for $(a+b)^5$, the coefficients for the sixth power of $a+b$ are

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1.$$

(1) $$\begin{aligned}(a+2b)^4 &\equiv a^4+4a^3(2b)+6a^2(2b)^2+4a(2b)^3+(2b)^4\\ &\equiv a^4+8a^3b+24a^2b^2+32ab^3+16b^4.\end{aligned}$$

(2) $$\begin{aligned}(2x-y)^5 &\equiv (2x)^5+5(2x)^4(-y)+10(2x)^3(-y)^2\\ &\quad +10(2x)^2(-y)^3+5(2x)(-y)^4+(-y)^5\\ &\equiv 32x^5-80x^4y+80x^3y^2-40x^2y^3+10xy^4-y^5.\end{aligned}$$

EXERCISES.

1. Extend Pascal's triangle to include the expansion of $(a+b)^{10}$.

Write out the following expansions:

2. $(a+b)^8$. (Use Pascal's triangle.)

3. $(x+y)^7$.

4. $(m+n)^8$.

5. $(1+x)^9$.

6. $(y+1)^{10}$.

7. $(2x+a)^4$.

8. $(p-q)^8$.

9. $(3y-2x)^4$.

10. $(2a+b)^5$.

11. $(x-3y)^5$.

12. $(21x+a)^6$.

13. $(x^2-2y)^6$.

14. $(x^2-y^2)^9$.

15. $\left(x+\dfrac{y}{2}\right)^8$.

16. $(3x-4y)^7$.

17. $[(x+y)+3]^4$.

18. $[(a+b)-2x]^4$.

19. $[3a-(x-y)]^5$.

20. $\left(\dfrac{x}{5}-\dfrac{2y}{3}\right)^6$.

EXERCISES.

By comparison with types *write down* the following products:

1. $(x+5)(x+3)$.

2. $(x+10)(x-2)$.

3. $(x-5)(x+8)$.

4. $(ax+3)(ax+5)$.

5. $(xy+a)(xy-b)$.

6. $(4x+2)(4x-5)$.

7. $(x^2+5)(x^2-10)$.

8. $(\overline{x+a}+3)(\overline{x+a}-7)$.

9. $(3 + by)(4 + by)$.

10. $(\overline{x^2 + 3x} + 2)(\overline{x^2 + 3x} - 5)$.

11. $(3x + y)(3x - 2y)$.

12. $(a + b + 12)(a + b - 6)$.

13. $\left(x - 5 + \dfrac{y}{2}\right)\left(x - 5 - \dfrac{y}{2}\right)$.

14. $(3x + y)(3x - y)$.

15. $(2x - 4y)(2x + 4y)$.

16. $(\overline{2x + y} + 8)(\overline{2x + y} - 8)$.

17. $(\overline{x^2 + y^2} - 4)(\overline{x^2 + y^2} + 4)$.

18. $(ax + by + c)(ax + by - c)$.

19. $(x^2 + 1 + x)(x^2 + 1 - x)$.

20. $(xy + yz + zx - a)(xy + yz + zx + a)$.

Perform the operations indicated:

21. $(x + y + z)^2$.

22. $(3x + 2y + z)^2$.

23. $(ax + by + c)^2$.

24. $(x + 3)^2 - 12x$.

25. $(x + 3y - 4)^2 + 16(x + 3y)$.

26. $3x + 2y + 6)^2 - 24(3x + 2y)$.

27. $(x^n + y^n)^2 - 4x^n y^n$.

28. $(x + y + z + w)^2$.

29. $(x + 3)(x + 5)(x + 6)$.

30. $(x + 3)(x - 3)(x + 5)(x - 5)$.

31. $(x + 3a)(x^2 - 3ax + 9a^2)$.

32. $(x + 3y)^3$.

33. $(x + y)^3 - 3xy(x + y)$.

34. $(a^2 + b^2 + c^2 - ab - bc - ca)(a + b + c)$.

35. $(x^2 + 4y^2 + 1 - 2xy - 2y - x)(x + 2y + 1)$.

Show the truth of the following identities:

36. $(x + a)^2 - (x - a)^2 \equiv 4ax$.

37. $(x + a)^2 - 4ax \equiv (x - a)^2$.

38. $(x^2 + xy + y^2)(x - y) \equiv x^3 - y^3$.

39. $(x^2 - xy + y^2)(x + y) \equiv x^3 + y^3$.

40. $(x^3 + x^2y + xy^2 + y^3)(x - y) \equiv x^4 - y^4$.

41. $(x^3 - x^2y + xy^2 - y^3)(x + y) \equiv x^4 - y^4$.

42. $(x^2 + xy + y^2)(x^2 - xy + y^2) \equiv x^4 + x^2y^2 + y^4$.

43. $(x^2 + y^2 + z^2 - xy - yz - zx)(x + y + z) \equiv x^3 + y^3 + z^3 - 3xyz$.

70. Division Identities. From the relation of division to multiplication, every multiplication identity gives rise to at least two division identities. The following division identities are of importance:

1. $[x^2+(a+b)x+ab]\div(x+a)\equiv x+b.$
2. $(x^2+2\,ax+a^2)\div(x+a)\equiv x+a.$
3. $(x^2-2\,ax+a^2)\div(x-a)\equiv x-a.$
4. $(x^2-a^2)\div(x-a)\equiv x+a.$
5. $(x^3-y^3)\div(x-y)\equiv x^2+xy+y^2.$
6. $(x^3+y^3)\div(x+y)\equiv x^2-xy+y^2.$
7. $x^4-y^4\div(x-y)\equiv x^3+x^2y+xy^2+y^3.$

It should be noted that in each of these identities, the quotient might be the divisor, and the divisor the quotient.

(1) $(x^2-64)\div(x+8)=?$

This is an example of type 4. By a comparison with that type we see at once that the quotient is $x-8$.

(2) $(x^2-11\,x+30)\div(x-6)=?$

This is an example of type 1.

$$a=-6;\ ab=30;\ \text{hence, } b=-5.$$

The quotient is $x-5$.

(3) $(27-a^3)\div(3-a)=?$

If we notice that $27=3^3$, we see that this is of the form of type 5.

Hence, $(27-a^3)\div(3-a)=9+3\,a+a^2$.

EXERCISES.

Perform the following divisions by comparison with type forms:

1. $(a^2-2\,a-63)\div(a+7)$.
2. $[(2\,x)^4-b^4]\div(4\,x^2-b^2)$.

3. $(9 + 6a + a^2) \div (3 + a)$.

4. $(8a^3 + b^3) \div (2a + b)$.

5. $(81a^2 - 25b^4) \div (9a + 5b^2)$.

6. $(1 - 10a + 25a^2) \div (1 - 5a)$.

7. $(y^2 + 11y - 26) \div (y + 13)$.

8. $[(a + b)^3 - 64] \div (a + b - 4)$.

9. $[27a^3 - (x - y)^3] \div [3a - (x - y)]$.

10. $[(x + y)^2 - 4a^2b^2] \div (x + y + 2ab)$.

11. $(x^4 - 14x^2 - 51) \div (x^2 - 17)$.

12. $[(x + y)^2 + 11(x + y) - 60] \div (x + y - 4)$.

13. $[(x + y)^2 - 16(x + y)(a + b) + 48(a + b)^2] \div [(x + y) - 4(a + b)]$.

14. $(x^{2n} - 9x^n - 112) \div (x^n - 16)$.

15. $(y^{2n} + 16y^n + 64) \div (y^n + 8)$.

16. $[(a + b)^2 - 6x(a + b) + 9x^2] \div (a + b - 3x)$.

17. $(x^{4n} - 25y^{2n}) \div (x^{2n} + 5y^n)$.

18. $[(x + y)^{2n} - (a + b)^{4n}] \div [(x + y)^n - (a + b)^{2n}]$.

19. $[(ax + b)^3 + 125x^3] \div [(ax + b)^2 - 5x(ax + b) + 25x^2]$.

20. $[(ax^2 + bx + c)^2 - (lx + m)^2] \div (ax^2 + bx + c + lx + n)$.

21. $(y^{3n} - x^{3n}) \div (y^n - x^n)$.

22. $y^{4n} - x^{4n} \div (y^n - x^n)$.

23. $[(\sqrt{3x + y})^2 - a^2] \div (\sqrt{3x + y} - a)$.

24. $[(\sqrt{ax + b})^3 - 64y^3] \div (\sqrt{ax + b} - 4y)$.

25. $[(\sqrt{ax + by})^3 + 125a^6] \div (\sqrt{ax + by} + 5a^2)$.

CHAPTER VI.

FACTORING.

71. Products, Factors. *Numbers which are multiplied together to form a product are called factors of that product.*

For example, in $5 \times a \times b \equiv 5\,ab$, 5, a, b are *factors* of the *product* $5\,ab$. Only expressions free from divisions and roots will be considered in factoring. In $7\,a(b+c)(x+y+z)$, 7 is a *numerical factor* or *numerical multiplier*, a is a *monomial factor*, $b+c$ is a *binomial factor*, and $x+y+z$ is a *trinomial factor*.

In multiplication, we have the factors given to find the product; in factoring, the product is given to find the factors.

72. The Degree and Number of Factors. *The degree of an algebraic monomial is the number of letters composing it.*

Thus, a^2b is an expression of the *third* degree, being made up of the product of $a \times a \times b$. a^2x^2y is an expression of the *fifth* degree in a, x and y; it is an expression of the second degree in a, also in x; it is of the first degree in y.

The degree of an algebraic polynomial is the highest number of letters found in any term.

Thus, $x^2 + 3\,x + 4$ is an expression of the second degree, containing $x \times x$ in its highest term. $ax^2 + bx$ is an expression of the third degree in a and x, but is an expression of the second degree in x.

The number of factors of an algebraic expression is not greater than the degree of the expression.

EXERCISES.

Determine the degree of the following with regard to all the letters:

1. $x^2 + ax$.
2. $x + y + z$.
3. $x^2 + xy + y^2$.
4. $x^3 + 3x^2 + 4x + 16$.
5. $ax^2 + bx + c$.
6. $x^3 + y^3 + z^3 - 3xyz$.
7. $x^2 - a^2$.
8. $x^2 + 2ax + a^2$.
9. $abc + a^2 + b^2 + a^2b^2$.
10. $a^2bc + b^2ca + c^2ab$.

73. Monomial Factors. Factors of monomial expressions may be written down by inspection.

Thus, $$a^2bx^2 \equiv a \cdot a \cdot b \cdot x \cdot x;$$
also, $$5x^2y^3 \equiv 5 \cdot x \cdot x \cdot y \cdot y \cdot y.$$
Here, 5 is not an algebraic factor in the sense of determining degree; it is a *numerical multiplier*.

Monomial factors contained in polynomials may be seen as the inverse of the distributive law.

$$(a + b + c)m \equiv am + bm + cm.$$

Reverse this identity, and we have the factors of

$am + bm + cm$, namely m and $a + b + c$.

EXERCISES.

Factor:

1. $14a^2x^4y$.
2. $-3xy^2z$.
3. $\frac{2}{3}x^2yz^2$.
4. $\frac{5}{6}a^2b^2c^3$.
5. $-10a^3bc$.
6. $12x^3yz^2$.
7. $4a^2bc^4$.
8. $4x^2 + 7x \equiv x(4x + 7)$.
9. $ax^2 + ay$.
10. $3ax + 6a^2x^2$.
11. $5a^2 + 10ab + 5abc$.
12. $3xy - 6x^2y^2 + 7x^3y^3$.

13. $3\,mx + 4\,m^2x^2 + 2\,mxy.$

14. $-2\,lx + 4\,l^2y + 6\,ly.$

15. $axy + a^2xy + 3\,ax^2y^2.$

16. $3\,a^2xy - 15\,ax^2y + 21\,a^3xy^3.$

17. $5\,x^3yz + 30\,x^2y^2z - 40\,xyz^3.$

18. $7\,a^3x^3yz + 7\,a^3xy^3z + 7\,a^3xyz^3.$

19. $3a^2bc\sqrt{x+y} - 6\,ab^2c\sqrt{x-y}.$

20. $5\,x^2yz\sqrt{ax+b}$
$+10\,xy^2z\sqrt{ax-b}.$

I. TYPES.

A great number of algebraic expressions may be factored by comparison with some *known form* of product. The identities of the preceding chapter are reversible; when so written they become *Types* in *factoring*.

74. The Type $x^2 - a^2$, the Difference of Two Squares. This expression is recognized as the product of $x + a$ by $x - a$. Hence,

$$x^2 - a^2 \equiv (x + a)(x - a).$$

The difference of two squares equals the product of the sum and difference of the square roots of the two numbers.

Thus, $$x^2 - 16 \equiv (x + 4)(x - 4);$$

also, $$(a + b)^2 - 16 \equiv (a + b + 4)(a + b - 4).$$

EXERCISES.

Factor:

1. $a^2 - 4\,b^2.$
2. $4\,a^2 - 9\,b^2.$
3. $a^2b^2 - c^2.$
4. $16\,a^2b^2 - 25\,c^2.$
5. $4 - 9\,x^2.$
6. $25\,x^2y^2 - 36\,z^2w^2.$
7. $(x + y)^2 - a^2.$
8. $(x + 3\,y)^2 - z^2.$
9. $(3\,x - 2\,y)^2 - (z + a)^2.$
10. $4\,(x + 2\,y)^2 - 9\,(a + b)^2.$
11. $x^4 - y^4 \equiv (x^2 - y^2)(x^2 + y^2) \equiv (x + y)(x - y)(x^2 + y^2).$

In Exercise 11 we have two first-degree factors, $x+y$ and $x-y$, also the second-degree factor, x^2+y^2. No factors of x^2+y^2 can be found unless radicals be employed. Such a factor may be called *irreducible.*

Factor:

12. $4x^2-16(x-34)^2$.

13. $16-a^4$.

14. $(x+y)^4-4x^2y^2$.

15. $(3x+4y+5)^2-9$.

16. $(-3x+2y-5)^2-(x+y)^2$.

17. $(lx+my+n)^2-4(ax+by+c)^2$.

18. $(ax+by)^2-4(lz+mw)^2$.

19. $(a^nx^n)^2-(b^ny^n)^2$.

20. $x^{2n}y^{2n}-z^{2n}w^{2n}$.

21. $(x+1)^2-a^2$.

22. $(x+3)^2-4y^2$.

23. $(3x+a)^2-9b^2$.

24. $(x+a)^4-(a+b)^4$.

25. $(x+3y)^2-(3x+y)^2$.

75. The Type $x^2+(a+b)x+ab$. This expression is the product obtained by multiplying $x+a$ by $x+b$. Hence,

$$x^2+(a+b)x+ab\equiv(x+a)(x+b).$$

Examples belonging to this type assume the form

$$x^2+sx+p,$$

where p is the algebraic product of two numbers, and s is their algebraic sum.

If s and p be integers, the factors of p may sometimes be found by inspection such that their sum shall be s.

Thus, to factor x^2+6x+8, we must find two factors of 8 whose sum is 6. These are seen to be 4 and 2. Hence,

$$x^2+6x+8\equiv(x+4)(x+2).$$

To factor $a^2+10a-24$, we must find two factors of -24 whose sum is 10. These are 12 and -2. Hence,

$$a^2+10a-24\equiv(a-2)(a+12).$$

It should be observed *that if in x^2+sx+p, p be positive, the factors of p chosen must be of like signs, and if s be positive, both factors of p must be positive. If s be negative, both two factors of p must be negative; if p be negative, one factor of p must be positive and the other negative, and the sign of s shows which is numerically the larger.*

EXERCISES.

1. Factor $y^2-5y-24$.

Here the factors of -24 are $-1, 24$; $-2, 12$; $-3, 8$; $-4, 6$; and also these numbers with their signs changed. A pair of factors must be chosen whose sum is -5; this is seen to be $3, -8$. Hence,

$$y^2-5y-24 \equiv (y+3)(y-8).$$

2. Factor $(x+2)^2-5(x+2)-14$.

The two factors of -14, whose sum is -5, are -7 and 2. Hence,

$$\begin{aligned}(x+2)^2-5(x+2)-14 &\equiv (\overline{x+2}+2)(\overline{x+2}-7)\\ &\equiv (x+4)(x-5).\end{aligned}$$

3. Factor $(x^2+3x)^2-8(x^2+3x)-20$.

The factors of -20, whose sum is -8, are -10 and 2. Hence,

$$\begin{aligned}(x^2+3x)^2-8(x^2+3x)-20 &\equiv (x^2+3x+2)(x^2+3x-10)\\ &\equiv \{(x+1)(x+2)\}\{(x+5)(x-2)\}.\end{aligned}$$

Factor:

4. x^2+3x+2.

5. x^2-3x+2.

6. x^2+x-2.

7. x^2-x-2.

8. $a^2x^2+5ax+6$.

9. $x^2+5xy+6y^2$.

10. $x^2y^2-3xy-10$.

11. $1+3xy-10x^2y^2$.

12. $6x^2-5xy+y^2$.

13. $(x+y)^2+9(x+y)+20$.

14. $(a+3b)^2-(a+3b)-20.$

15. $(6x+3y)^2+10(6x+3y)+16.$

16. $(x^2+y^2)^2-6(x^2+y^2)-27.$

17. $(2x+3y+5)^2+9(2x+3y+5)+18.$

18. $(x^2+5x)^2+10(x^2+5x)+24.$

19. $x^4-13x^2+36.$

20. $(ax+by)^2+8(ax+by)+7.$

21. $(ax+by)^2-(l+m)(ax+by)+lm.$

22. $(x^2+6x)^2+17(x^2+6x)+72.$

23. $(x^2-5x+4)^2-(x^2-5x+4)-2.$

24. $x^{2n}-10x^n+16.$

25. $(ax)^{2n}+(l+m)(ax)^n+lm.$

76. The Types $x^2+2ax+a^2$ and $x^2-2ax+a^2$. These two trinomials are perfect squares of $x+a$ and $x-a$, respectively.

$$x^2+2ax+a^2 \equiv (x+a)(x+a) \equiv (x+a)^2.$$
$$x^2-2ax+a^2 \equiv (x-a)(x-a) \equiv (x-a)^2.$$

The sum of the squares of two numbers, increased (or diminished) by twice the product of the numbers, equals the square of the sum (or difference) of the two numbers.

EXERCISES.

1. Factor x^2+6x+9.

Here x^2 and 9 are the squares of x and 3, and $6x$ is twice the product of 3 and x; hence,

$$x^2+6x+9 \equiv (x+3)(x+3) \equiv (x+3)^2.$$

2. Factor $9x^2+6x+1$.

$$9x^2+6x+1 \equiv (3x+1)(3x+1) \equiv (3x+1)^2.$$

3. Factor $16\,a^2 + 40\,ab + 25\,b^2$.

$$16\,a^2 + 40\,ab + 25\,b^2 \equiv (4\,a)^2 + 2\,(4\,a)\,(5\,b) + (5\,b)^2$$
$$\equiv (4\,a + 5\,b)\,(4\,a + 5\,b) \equiv (4\,a + 5\,b)^2.$$

4. Factor $y^2 - 16\,yz + 64\,z^2$.

$$y^2 - 16\,yz + 64\,z^2 \equiv y^2 - 2\,(y)\,(8\,z) + (8\,z)^2$$
$$\equiv (y - 8\,z)\,(y - 8\,z) \equiv (y - 8\,z)^2.$$

5. Factor $(x^2 + 4\,x)^2 - 4\,(x^2 + 4\,x) + 4$.

In this expression we may consider $x^2 + 4\,x$ as a single quantity.

$$(x^2 + 4\,x)^2 - 4\,(x^2 + 4\,x) + 4 \equiv (x^2 + 4\,x)^2 - 2\,(x^2 + 4\,x)\,(2) + (2)^2$$
$$\equiv (x^2 + 4\,x - 2)\,(x^2 + 4\,x - 2)$$
$$\equiv (x^2 + 4\,x - 2)^2.$$

Factor:

6. $x^2 - 8\,x + 16$.

7. $4\,x^2 - 12\,x + 9$.

8. $a^2x^2 + 10\,axy + 25\,y^2$.

9. $49\,a^2b^2 - 14\,ab + 1$.

10. $100 - 20\,ab + a^2b^2$.

11. $(ax + b)^2 + 2\,c\,(ax + b) + c^2$.

12. $(3\,x + 4\,y)^2 - 6\,(3\,x + 4\,y) + 9$.

13. $4\,x^2 + 9\,y^2 - 12\,xy$.

14. $(ax + by + c)^2 + 8\,(ax + by + c) + 16$.

15. $(x^2 + y^2)^2 - 2\,(x^2 + y^2)\,z^2 + z^4$.

The next three exercises are squares of trinomials. See page 73, 5.

16. $x^2 + y^2 + z^2 + 2\,xy + 2\,yz + 2\,zx$.

17. $a^2 + b^2 + c^2 - 2\,ab + 2\,bc - 2\,ca$.

18. $4\,x^2 + 9\,y^2 + z^2 + 12\,xy + 6\,yz + 4\,zx$.

19. $x^{2n}+12x^n+36$.

20. $y^{4n}-14y^{2n}+49$.

21. $a^{2n}+12a^nb^{2n}+36b^{4n}$.

22 $(x+y)^{2n}-6a(x+y)^n+9a^2$.

77. The Types x^3-y^3 and x^3+y^3. It has been shown by actual multiplication that

$$x^3-y^3\equiv(x-y)(x^2+xy+y^2),$$

and

$$x^3+y^3\equiv(x+y)(x^2-xy+y^2).$$

EXERCISES.

1. Factor $8a^3-27b^3$.

$$\begin{aligned}8a^3-27b^3&\equiv(2a)^3(3-b)^3\\&\equiv(2a-3b)[(2a)^2+(2a)(3b)+(3b)^2],\\&\equiv(2a-3b)(4a^2+6ab+9b^2).\end{aligned}$$

2. Factor $(x^2+y^2)^3-8x^3y^3$.

$$\begin{aligned}(x^2+y^2)^3-8x^3y^3&\equiv(x^2+y^2)^3-(2xy)^3\\&\equiv[(x^2+y^2)-2xy][(x^2+y^2)^2+(x^2+y^2)2xy\\&\qquad+(2xy)^2]\\&\equiv(x-y)^2[(x^2+y^2)^2+2xy(x^2+y^2)+4x^2y^2].\end{aligned}$$

3. Factor $(x+5)^3+8b^3$.

$$\begin{aligned}(x+5)^3+8b^3&\equiv(x+5)^3+(2b)^3\\&\equiv[(x+5)+2b][(x+5)^2-(x+5)2b+(2b)^2]\\&\equiv[x+5+2b][(x+5)^2-2b(x+5)+4b^2].\end{aligned}$$

Factor:

4. x^3-8b^3.

5. $8a^3-27b^3$.

6. a^3x^3+64.

7. $a^3x^3-27y^3z^3$.

8. $(x+y)^3-125z^3$.

9. $(3x+4)^3+8y^3$.

10. $(x^2+3x+4)^3-64$.

11. $(ax+by)^3-c^3z^3$.

12. $(3x+4y)^3+(2x+y)^3$.

Certain expressions may be transformed into the sum or difference of two cubes, and the factors then found.

13. Factor $x^6 + y^6$.

In this case we may write

$$\begin{aligned} x^6 + y^6 &\equiv (x^2)^3 + (y^2)^3 \\ &\equiv [x^2 + y^2][(x^2)^2 - (x^2y^2) + (y^2)^2] \\ &\equiv [x^2 + y^2][x^4 - x^2y^2 + y^4]. \end{aligned}$$

14. Factor $x^6 - y^6$.

As in Exercise 13, we may write

$$\begin{aligned} x^6 - y^6 &\equiv (x^2)^3 - (y^2)^3 \\ &\equiv (x^2 - y^2)[(x^2)^2 + x^2y^2 + (y^2)^2] \\ &\equiv (x+y)(x-y)[x^4 + x^2y^2 + y^4] \\ &\equiv (x+y)(x-y)(x^2 + xy + y^2)(x^2 - xy + y^2). \end{aligned}$$

(See Exercise 42, page 80.)

Of course, the factors of $x^6 - y^6$ could have been obtained by comparing with the type $x^2 - a^2$. Thus,

$$\begin{aligned} x^6 - y^6 &\equiv (x^3)^2 - (y^3)^2 \\ &\equiv (x^3 + y^3)(x^3 - y^3) \\ &\equiv (x+y)(x^2 - xy + y^2)(x-y)(x^2 + xy + y^2). \end{aligned}$$

Other types of binomials, such as $a^4 - b^4$, $a^5 \pm b^5$, and so on, may appear for factoring, but such special cases will not be considered at this time.

Factor:

15. $8a^3 - b^6$.

16. $(ax)^6 + (by)^6$.

17. $x^6 - 1$.

18. $a^6x^6 + (y+z)^6$.

19. $(ax + by)^6 - (cz)^6$.

20. $64a^6 + (bc)^6$.

21. $[(x+y)^2]^3 - [(x-y)^2]^3$.

22. $(a^3 + 3a^2b + 3ab^2 + b^3)^2 + c^6$.

78. The Type $Ax^2 + Bxy + Cy^2$ or $Ax^2 + Bx + C$. If the first of these expressions is capable of separation into factors free of radicals, it must be composed of two binomials of the form $ax + by$ and $lx + my$, where a, b, l, m are algebraic numbers, *i.e.*, they may be $+$ or $-$, integral or fractional.

Multiplying these supposed factors together, we have

$$(ax + by)(lx + my) \equiv alx^2 + (am + bl)xy + bmy^2.$$

Hence, to factor $Ax^2 + Bxy + Cy^2$ we are to find four numbers, a, b, l, m, such that

$$al = A,\ bm = C,\ \text{and}\ am + bl = B.$$

This method is illustrated by the following examples:

(1) Factor $6x^2 + 31xy + 35y^2$.

The factors of 6 are 6, 1 and 3, 2; the factors of 35 are 35, 1, and 7, 5. A good plan is to arrange the letters thus,

$$\left.\begin{matrix} x + y \\ x + y \end{matrix}\right\}.$$

Now attach the factors of 6 and 35 to x, y, respectively, as a trial arrangement. Let us place them thus,

$$\left.\begin{matrix} 3x + 7y \\ 2x + 5y \end{matrix}\right\}.$$

The square terms appear correctly, $6x^2$, $35y^2$, but the *cross products*, $3x \cdot 5y$ and $2x \cdot 7y$, do not add so as to give $31xy$. Hence, our trial arrangement is not correct. Let us try

$$\left.\begin{matrix} 2x + 7y \\ 3x + 5y \end{matrix}\right\}.$$

This gives $6x^2 + (2 \times 5 + 3 \times 7)xy + 35y^2$, the correct product.

Hence, $6x^2 + 31xy + 35y^2 \equiv (2x + 7y)(3x + 5y)$.

The case in which y is equal to unity, giving $Ax^2 + Bx + C$, requires no special mention when the factors may be found by inspection as above.

(2) Factor $3x^2 + 16xy + 5y^2$.

In this example we are to find the arrangements of the factors of 3 and 5 with the letters x, y in

$$\left.\begin{array}{r} x+y \\ x+y \end{array}\right\},$$

such that the sum of the cross products shall be $16xy$.

By trial, the arrangement is seen to be

$$\left.\begin{array}{r} x+5y \\ 3x+y \end{array}\right\}.$$

Hence, $\quad 3x^2 + 16xy + 5y^2 \equiv (x+5y)(3x+y)$.

(3) Factor $15x^2 + 58x + 11$.

The factors of 15 are 3, 5 and 1, 15; the factors of 11 are 1, 11. Our trial arrangement may be

$$\left.\begin{array}{r} x+11 \\ 15x+1 \end{array}\right\}.$$

But the cross products $11 \times 15 + 1 \times 1$ do not give 58. Another trial may be

$$\left.\begin{array}{r} 3x+11 \\ 5x+1 \end{array}\right\},$$

which gives for the middle term $3x + 55x = 58x$.

Hence, $\quad 15x^2 + 58x + 11 \equiv (3x+11)(5x+1)$.

EXERCISES.

Factor:

1. $3x^2 + 7xy + 2y^2$.

2. $6x^2 - 5xy - 6y^2$.

3. $21x^2 + 31xy + 4y^2$.

4. $12x^2 + 2xy - 2y^2$.

5. $3ax^2 + (9+a)bx + 3b^2$.

6. $20y^2 - 22yz + 6z^2$.

7. $24y^2 - 26yz - 8z^2$.

79. The Type $ax^2 + bx + c$. This expression is called the *General Quadratic* in a single variable x. For different values of a, b, c this expression represents every quadratic that may be written. Thus, with $a = 5$, $b = 6$, $c = 2$, we have $5x^2 + 6x + 2$; with $a = 4$, $b = -5$, $c = -1$, we have $4x^2 - 5x - 1$, etc.

We shall illustrate the method of factoring the general quadratic by a few special examples.

(1) Factor $2x^2 + 16x - 20$.

By dividing out 2, we get

$$\begin{aligned} 2x^2 + 16x - 20 &\equiv 2(x^2 + 8x - 10) \\ &\equiv 2[x^2 + 8x + 16 - 10 - 16], \text{ adding and subtracting 16.} \\ &\equiv 2[(x+4)^2 - 26], \qquad \text{rearranging terms.} \end{aligned}$$

26 may be written $(\sqrt{26})^2$, and we have

$$\begin{aligned} 2x^2 + 16x - 20 &\equiv 2[(x+4)^2 - (\sqrt{26})^2], \qquad \text{the difference of two squares.} \\ &\equiv 2[x + 4 + \sqrt{26}]\cdot[x + 4 - \sqrt{26}]; \text{ the product of sum and difference.} \end{aligned}$$

NOTE. Adding 16 is called *Completing the Square* of the first two terms. To complete the square of $x^2 + 2mx$ we must add m^2; *i.e., we add the square of half the coefficient of x.* To complete the square of $x^2 + 6x$ we add $\left(\frac{6}{2}\right)^2 \equiv 3^2$. To complete the square of $x^2 + kx$ we add $\left(\frac{k}{2}\right)^2$.

Complete the square:

1. $x^2 + 5x$.

Here $(\frac{5}{2})^2$ must be added.

If we wish the expression to remain unchanged, we must also subtract $(\frac{5}{2})^2$.

Hence, $x^2+5x \equiv x^2+5x+\frac{25}{4}-\frac{25}{4},$
$$\equiv (x+\tfrac{5}{2})^2-\tfrac{25}{4}.$$

2. $x^2-3x.$
3. $x^2+7x.$
4. $x^2-8x.$
5. $3x^2+9x \equiv 3(x^2+3x).$
6. $5x^2-25x.$
7. $11x^2+33x.$
8. $5x^2+7x.$
9. $3x^2-8x.$
10. $7x^2-35x.$
11. $9x^2-25x.$

(2) Factor $5x^2+15x-10.$

$5x^2+15x-10$

$\equiv 5[x^2+3x-2],$ dividing by 5.

$\equiv 5[x^2+3x+(\frac{3}{2})^2-(\frac{3}{2})^2-2],$ completing the square.

$\equiv 5[(x+\frac{3}{2})^2-\frac{9}{4}-2],$ rearranging.

$\equiv 5[(x+\frac{3}{2})^2-\frac{17}{4}],$

$\equiv 5[(x+\frac{3}{2})^2-(\sqrt{\frac{17}{4}})^2],$ difference of two squares.

$\equiv 5\left[x+\dfrac{3}{2}+\dfrac{\sqrt{17}}{2}\right]\cdot\left[x+\dfrac{3}{2}-\dfrac{\sqrt{17}}{2}\right],$ product of sum and difference.

(3) Factor $4x^2+6x+2.$

$4x^2+6x+2$

$\equiv 4[x^2+\frac{3}{2}x+\frac{1}{2}],$ dividing out 4.

$\equiv 4[x^2+\frac{3}{2}x+(\frac{3}{4})^2-(\frac{3}{4})^2+\frac{1}{2}],$ completing the square.

$\equiv 4[(x+\frac{3}{4})^2-\frac{1}{16}],$ rearranging.

$\equiv 4[(x+\frac{3}{4})^2-(\frac{1}{4})^2],$ rearranging.

$\equiv 4[x+\frac{3}{4}+\frac{1}{4}]\cdot[x+\frac{3}{4}-\frac{1}{4}],$ product of sum and difference.

$\equiv 4[x+1]\cdot[x+\frac{1}{2}].$

According to the method here illustrated we may factor the general quadratic.

$ax^2 + bx + c$

$$\equiv a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right], \qquad \text{dividing out } a.$$

$$\equiv a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right], \text{ completing square.}$$

$$\equiv a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right], \qquad \text{combining terms.}$$

$$\equiv a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2\right], \qquad \text{difference of two squares.}$$

$$\equiv a\left[x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right] \cdot \left[x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right].$$

Hence, to factor any quadratic in which a, b, c have been replaced by numerical values, we need merely to replace those letters in the general factors above by the special values found in the given example.

(4) Factor $3x^2 + 8x + 2$.

Here $a = 3$, $b = 8$, $c = 2$; hence, we have

$3x^2 + 8x + 2$

$$\equiv 3\left[x + \frac{8}{2 \times 3} + \frac{\sqrt{64 - 4 \cdot 3 \cdot 2}}{2 \cdot 3}\right]\left[x + \frac{8}{2 \cdot 3} - \frac{\sqrt{64 - 4 \cdot 3 \cdot 2}}{2 \cdot 3}\right]$$

$$\equiv 3\left[x + \frac{4}{3} + \frac{\sqrt{40}}{6}\right]\left[x + \frac{4}{3} - \frac{\sqrt{40}}{6}\right].$$

(5) Factor $5x^2 - 7x + 3$.

In this case $a = 5$, $b = -7$, $c = 3$; hence,

$5x^2 - 7x + 3$

$$\equiv 5\left[x + \frac{-7}{2 \cdot 5} + \frac{\sqrt{(-7)^2 - 4 \cdot 5 \cdot 3}}{2 \cdot 5}\right] \cdot \left[x + \frac{-7}{2 \cdot 5} - \frac{\sqrt{(-7)^2 - 4 \cdot 5 \cdot 3}}{2 \cdot 5}\right]$$

$$\equiv 5\left[x - \frac{7}{10} + \frac{\sqrt{-11}}{10}\right]\left[x - \frac{7}{10} + \frac{\sqrt{-11}}{10}\right].$$

NOTE. The factors in this case, involving the square root of a *negative* number are called imaginary. Such numbers as $\sqrt{-11}$, $\sqrt{-5}$, $\sqrt{-a^2}$, etc., do not belong to the algebraic number system explained in Chapter I.

(6) Factor $-3x^2+4x+2$.

Here $a=-3$, $b=4$, $c=2$; hence,

$$-3x^2+4x+2\equiv-3\left[x+\frac{4}{2\cdot(-3)}+\frac{\sqrt{4^2-4(-3)\cdot 2}}{2\cdot(-3)}\right]$$
$$\times\left[x+\frac{4}{2\cdot(-3)}-\frac{\sqrt{4^2-4(-3)\cdot 2}}{2\cdot(-3)}\right]$$
$$\equiv-3\left[x-\frac{2}{3}+\frac{\sqrt{40}}{-6}\right]\left[x-\frac{2}{3}-\frac{\sqrt{40}}{-6}\right]$$
$$\equiv-3\left[x-\frac{2}{3}-\frac{1}{6}\sqrt{40}\right]\left[x-\frac{2}{3}+\frac{1}{6}\sqrt{40}\right].$$

EXERCISES.

Factor:

1. $5x^2-25x-15$.
2. $-2x^2+14x-10$.
3. $3x^2-5x+9$.
4. $7x^2-35x+49$.
5. $-x^2+11x-3$.
6. $11x^2-55x+99$.
7. $-4x^2-28x+32$.
8. $6x^2-30x+48$.
9. $-8x^2+40x-8$.
10. $10x^2-70x+20$.

II. FACTORS BY REARRANGEMENT AND GROUPING OF TERMS.

The method of factoring certain algebraic expressions will often be suggested by a proper rearrangement or grouping of terms. Two general plans of grouping are worthy of attention.

80. Grouping with Regard to the Descending Powers of Some Letter.

(1) Factor $4x^2+y^2+z^2+4xy+4xz+2yz$.

If this example is not recognized as a perfect square, we should proceed thus:

$$\begin{aligned}4x^2+y^2+z^2+4xy+4xz+2yz &\equiv 4x^2+4x(y+z)+y^2+2yz+z^2\\ &\equiv 4x^2+4x(y+z)+(y+z)^2\\ &\equiv (2x+y+z)^2.\end{aligned}$$

(2) Factor $x^2+6ax-8bx+9a^2-24ab+16b^2$.

Arrange with regard to the letter x, giving

$$\begin{aligned}&x^2+6ax-8bx+9a^2-24ab+16b^2\\ &\qquad\equiv x^2+2x(3a-4b)+9a^2-24ab+16b^2\\ &\qquad\equiv x^2+2x(3a-4b)+(3a-4b)^2\\ &\qquad\equiv (x+3a-4b)^2.\end{aligned}$$

(3) Factor $x^2+7ax+6bx+10a^2+21ab+9b^2$.

Here we may arrange with regard to the letter a.

$$\begin{aligned}&x^2+7ax+6bx+10a^2+21ab+9b^2\\ &\qquad\equiv 10a^2+7a(x+3b)+x^2+6bx+9b^2\\ &\qquad\equiv 10a^2+7a(x+3b)+(x+3b)^2\\ &\qquad\equiv (5a+x+3b)(2a+x+3b).\end{aligned}$$

EXERCISES.

Factor:

1. $2a^2+3ax+x^2+7ab+3b^2+4bx$.
2. $2x^2-b^2+3y^2+bx-2by+7xy$.
3. $x^2+4y^2+9z^2-4xy-12yz+6xz$.
4. $4a^2+9b^2+16c^2+12ab-16ac-24bc$.
5. $2x^2-3y^2-3z^2-xy-5xz+10yz$.

81. Grouping with Regard to Some Letter that Enters in One Degree Only. By forming a product of a number of factors, one of which contains a letter not found in the others, we shall see the purpose and application of this method.

Let us multiply together the following:

$$(x+3a+b)(x+a+m).$$

The product is

$$x^2+4ax+3a^2+bx+ab+m(x+3a+b).$$

It will be noticed that m appears in but three terms, and that these terms when collected constitute m times the first factor of the product. Hence, *if a literal expression contain a single letter entering to a single degree only, the coefficient of that letter contains a factor of the given expression, if the expression has any factors.*

An example will illustrate this method.

Factor $3x^2+8xy+3kx+5ky+5y^2$.

Here k enters to a single degree.

Arrange with regard to k, and we have

$$3x^2+8xy+5y^2+(3x+5y)k.$$

If the expression can be factored, $3x+5y$ must be one of the factors; hence, $3x^2+8xy+5y^2$ must contain $3x+5y$ as a factor.

$$\begin{aligned}3x^2+8xy+5y^2+3kx+5ky &\equiv 3x^2+8xy+5y^2+k(3x+5y)\\ &\equiv (3x+5y)(x+y)+k(3x+5y)\\ &\equiv (3x+5y)(x+y+k).\end{aligned}$$

EXERCISES.

Factor: **1.** $4x^2+4xy-35y^2+5ay-2ax$.

2. $x^2+zx-4yz-xy-12y^2$.

3. $4ac+bc-b^2+12a^2-ab$.

4. $xy+4ay+x^2-ax-20a^2$.

5. $14x^2+7px-5xy+py-y^2$.

82. Binomial Factors by Trial. Let us divide x^2+6x+5 by $x-m$ in the ordinary way.

$$
\begin{array}{l}
x-m)x^2+6x+5(x+6+m, \qquad \text{quotient.} \\
\qquad\quad \underline{x^2-mx} \\
\qquad\quad (6+m)x+5 \\
\qquad\quad \underline{(6+m)x-m(6+m)} \\
\qquad\qquad\qquad m^2+6m+5, \text{ remainder.}
\end{array}
$$

It will be noticed that the remainder found on dividing x^2+6x+5 by $x-m$ is precisely the value that the dividend becomes when x has been replaced by m. Thus, if in x^2+6x+5, we put $x=m$, we get m^2+6m+5, the same as the remainder above.

This remainder is, of course, true for any value of m. We make a few special illustrations for various values of m.

(1) What is the remainder on dividing

$$x^2+6x+5 \text{ by } x-4?$$

Our result above shows the remainder to be the value of x^2+6x+5 when $x=4$; hence, the remainder after division is

$$4^2+6\times 4+5=45.$$

Verify by actual division.

(2) Find the remainder after division of

$$x^2+6x+5 \text{ by } x-5.$$

Replace x by 5, and the remainder is

$$5^2+6\times 5+5=60.$$

Verify by division.

(3) Find the remainder when x^2+6x+5 is divided by $x+5$. Here, $m=-5$; hence, the remainder found by dividing by $x+5$ is

$$(-5)^2+6(-5)+5=0.$$

The remainder being zero shows that $x+5$ is an exact divisor of x^2+6x+5.

(4) Find the remainder on dividing $x^2-8x+12$ by $x-8$.

The remainder is $(8)^2-8(8)+12=12$.

Hence, $x-8$ is not an exact divisor.

(5) Find the remainder on dividing $x^2-8x+12$ by $x-6$.

The remainder is $(6)^2-8(6)+12=0$.

Hence, $x-6$ is an exact divisor, *i.e.*, a factor.

In factoring by trial, the number of trials is limited to the number of divisors of the constant term.

(6) To find the factors of x^3-7x-6.

The factors of 6 are ± 1, ± 2, ± 3, ± 6.

When divided by $x-1$ the remainder is $1-7-6=-12$.

When divided by $x+1$ the remainder is $-1+7-6=0$.

When divided by $x-2$ the remainder is $8-14-6=-12$.

When divided by $x+2$ the remainder is $-8+14-6=0$.

When divided by $x-3$ the remainder is $27-21-6=0$.

No other divisors need be tried, for we already have the three $(x+1)$, $(x+2)$, and $(x-3)$, and an expression of the third degree can not have more than three factors.

EXERCISES.

Factor each of the following:

1. x^3-x^2-4x+4.
2. x^3+2x^2-4x-8.
3. $x^3+3x^2-13x-15$.
4. $x^3-6x^2+11x-6$.
5. $x^3-10x^2+31x-30$.
6. $x^3+2x^2-9x-18$.
7. $x^3+9x^2+26x+24$.
8. x^3+15x^2-x-15.
9. $x^3+12x^2+47x+60$.
10. $x^3-12x^2+48x-64$.

EXERCISES IN FACTORING.

1. $bx^2 - b.$
2. $10\,cm^2 - 40\,c^3.$
3. $x^2 + xy + xz + yz.$
4. $lm + mr - lr - r^2.$
5. $2\,x^2 + 3\,xy - 2\,xz - 3\,yz.$
6. $6 - 5\,a + a^2.$
7. $x^4 + x^2y^2 - 72\,y^4.$
8. $m^2 - 4\,m + 96.$
9. $ab^2 + 2\,a^2b + a^3.$
10. $3\,x^2 + 4\,x + 1.$
11. $2\,l^2 - 8\,(m+1)^2.$
12. $64\,a^4 - 81\,y^4.$
13. $27 - 64\,x^6.$
14. $16\,x^3y^2 - x^5.$
15. $a^3 - 2\,a^2 - a + 2.$
16. $m^4 - 4\,m^2n^2l^2 + 4\,n^4l^4.$
17. $(a+b+c)^2 - (a-b-c)^2.$
18. $14\,a^3 - 8\,a^2 - 21\,a + 12.$
19. $p^2 - 2\,pq + q^2 - r^2.$
20. $25\,x^2 - 10\,xy - 9\,z^2 + y^2.$
21. $9\,a^2 - 36\,ab + 36\,b^2.$
22. $a^2 + 29\,a + 120.$
23. $x^4 + 2\,x^2 - 99.$
24. $24\,l^2 + 18\,l + 3.$
25. $a^2(x-y) + 3\,a(y-x) + 2\,(x-y).$
26. $\frac{1}{81} - \frac{16\,x^2}{y^4}.$
27. $m^5 + m^3 - m^2 - 1.$
28. $a^2b^2c^2 - 2\,a^3b^2c^3 + a^4b^2c^4.$
29. $9\,x^2 - 6\,xy - 1 + y^2.$
30. $64\,b^4 - 12\,b^2 - 1.$

REVIEW EXERCISES.

1. Remove parentheses and simplify
$$2\,x - 3\,y - \{5\,x - [3\,y + 5\,x - (4\,x + y - \overline{3\,x - 4\,y})]\}.$$

2. Put $x = 5$ and $y = 1$ in the above exercise, and find the value.

3. From the sum of $3\,x - 8\,y + 2\,z$ and $5\,y - 7\,x - 3\,z$ take their difference.

4. Multiply out $(a^4+4)(a^2+2)(a^2-2)$.

5. Divide $x^{3n} - 3\,x^{2n}y + 3\,x^n y^2 - y^3$ by $x^n - y$.

6. Factor $ax^2 + bx^3 - a - bx$.

7. Simplify

$$(x+a)^2 - (x-a)^2 - [(x+a)^2 + (x-a)(x-a) - x^2].$$

8. Divide $x^4 + 4y^4$ by $x^2 - 2xy + 2y^2$.

9. Two numbers differ by 17. One third of the smaller is one greater than $\frac{1}{7}$ of the larger. What are the numbers? (Let $x =$ smaller, $x + 17 =$ larger.)

10. Divide $(x^3 + y^3)(x^3 - y^3)$ by $x^3 - 2x^2y + 2xy^2 - y^3$.

11. Factor $x^4 - 10x^2 + 9$.

12. Add with respect to x, $a^3x + a(b-c)x - y^3x + 11x$.

13. Find the value of

$$x + yz - y\{x^2 - (3y + xz)(5x + y - 6) - \overline{2x - y}\},$$

when $x = 5$, $y = 2$, and $z = 1$.

14. Find the value of $\dfrac{-x + \sqrt{3 - 2x^2}}{x(1+3x) - x^3}$, when $x = -\frac{1}{3}$.

15. $(2a^3 - 3x + 5)^2 =$ what?

16. Divide $x^{3m+1} - x^{2m}y^{n+1} + x^{m+1}y^{2n} - y^{3n+1}$ by $x^{2m} + y^{2n}$.

17. Multiply $a^{m+3}b + a^2b^{n+2}$ by $a^{m-2}b + ab^{n-2}$.

18. Multiply $(a^n + 1)(a^n - 1)(a^{2n} + 2a^n + 1)$.

19. Verify $a(a+1)(a+2)(a+3) \equiv (a^2 + 3a + 1)^2 - 1$.

20. From

$$(x+y+z)(x+y-z) \text{ take } x^2 - \{y^2 - [2y^2 - (-2xy + z^2)]\}.$$

21. Find the sum, difference, product, and quotient of

$$4x(y-z)^{2m-4} \text{ and } x(y-z)^{2m-4}.$$

22. Factor $a^3 - 4a^2 + a + 6$.

23. Divide $8x^3 - y^3 + z^3 + 6xyz$ by $y - z - 2x$.

24. From $\frac{1}{3}(x - 3y) - \frac{1}{2}(9y - 2x)$ take $\frac{1}{12}(7x - 9y)$.

25. Divide $\frac{1}{3}a^3 - \frac{17}{36}a^2 + \frac{1}{3}a - \frac{1}{8}$ by $\frac{2}{3}a - \frac{1}{2}$.

CHAPTER VII.

DIVISORS AND MULTIPLES.

83. Highest Common Factor.

A Common Factor of two or more numbers is a factor of each of them.

a is a common factor of ax, a^2y, and ab^3. $x - y$ is a common factor of $x^2 - y^2$ and $x^3 - y^3$.

The Highest Common Divisor of two or more numbers is the product of all their common factors.

a^2x is the Highest Common Divisor (H.C.D.) of a^3xy, $3a^2x^2$, and $5a^2xz$. a^2 is common, and so is x.

In arithmetic, the term *Greatest Common Divisor* is frequently used. This term is not applicable in algebra. In the above example, a^2 may or may not be greater than a. If a is less than 1, then a^2 is less than a. Hence, in algebra the term *Highest Common Divisor* is used.

84. Highest Common Divisor of Monomials. RULE. *To the Greatest Common Divisor of the numerical coefficients affix each letter common to all the monomials, and to the lowest power it occurs in any one of them.*

$$3a^2x^4y^3z^6,\ 4a^4x^3y^5z^2,\ 5a^3x^5y^3z.$$

The G.C.D. of 3, 4, and 5 is 1.

The common letters with the proper exponents are a^2, x^3, y^3, z.

Hence, the required H.C.D. is $1a^2x^3y^3z$.

EXERCISES.

Find the H. C. D. of:

1. $4x^3y^2z,\ 8xy^5z^2,\ 12abyz^2.$
2. $5a^3bc^2,\ 10a^2b^3xy,\ 25a^2b^4cx.$
3. $16a^3x^3y^4z,\ 48a^4bx^5y^2z^4,\ 36a^5b^4x^3y^7z^6.$
4. $14a^3b^2c^4m^3n^2,\ 21a^4b^4m^4,\ 42a^5b^2m^6n^8.$
5. $22p^3q^4x^5y,\ 44p^4q^2x^3y^5,\ 66p^2q^8x^7y^8.$
6. $14(a-b)^2x^3y,\ 18(a-b)^3x^2y^3,\ 12(a-b)^2x^4y^2.$
7. $15(x-y)^3z^5,\ 21(x-y)^4z^8,\ 33(x-y)^6z^{10}.$
8. $18(a^2-b^2)^2xy,\ 27(a^2-b^2)^3x^2y,\ 36(a^2-b^2)^2x^4y^2.$

85. Highest Common Divisor of Polynomials. The H.C.D. of polynomials may be found by factoring. If each factor is considered as a single quantity, the method is the same as used in finding the H. C. D. of monomials.

(1) The H. C. D. of x^2-8x+7, x^2-1, x^2+3x-4 is found as follows:

$$\begin{aligned} x^2-8x+7 &\equiv (x-7)(x-1), \\ x^2-1 &\equiv (x+1)(x-1), \\ x^2+3x-4 &\equiv (x+4)(x-1). \end{aligned}$$

It is seen at once that $x-1$ is the H. C. D.

(2) Find the H. C. D. of

$$x^3+5x^2-14x,\ x^4-8x,\ x^4-4x^3+4x.$$

$$\begin{aligned} x^3+5x^2-14x &= x(x+7)(x-2), \\ x^4-8x &= x(x^2+2x+4)(x-2), \\ x^4-4x^3+4x^2 &= x^2(x-2)(x-2). \end{aligned}$$

Here we see that $x(x-2)$ is the H. C. D.

The H. C. D. is sometimes used in reducing fractions to their lowest terms.

EXERCISES.

Find the H. C. D. of:

1. $x^2 - y^2,\ x^2 - 2xy + y^2,\ x^2 - xy.$
2. $a^2 - b^2,\ a^3 - ab^2,\ a^2 + 2ab + b^2.$
3. $x^2 - y^2,\ x^3 + y^3,\ x^3 + x^2y.$
4. $x^2 - 7x + 12,\ x^2 + 2x - 15,\ x^2 - 9.$
5. $a^2 + 8a + 15,\ a^2 - 2a - 35,\ a^2 + 3a - 10.$
6. $b^2 - 14b + 49,\ b^2 + b - 56,\ b^2 - b - 42.$
7. $3x^2 - 12x + 12,\ (x-2)^3,\ 3x^2 - 12.$
8. $a^2b^2 - b^4,\ ab^2 + b^3,\ ab - b^2.$
9. $x^6 - x^3y^3,\ x^2(xy - y^2)^2.$
10. $(x^3 - 3x^2)^2,\ x^5 - x^4 - 6x^3.$
11. $x^6 - 2x^5 - 35x^4,\ x^6 - 25x^4.$
12. $x^2 + 2xy + y^2,\ x^3 + y^3 + 3xy(x+y).$
13. $m^2 - 3m - 70,\ m^3 - 11m^2 + 10m.$
14. $x^2 - xy + xz - yz,\ xy - y^2.$
15. $a^3 - 8,\ a^4b^2 - 4a^2b^2,\ 4a^2 - 16a + 16.$

86. Lowest Common Multiple.

When two algebraic expressions are so related that the first is an exact divisor of the second, the second is said to be a multiple of the first.

$6a^2bcx$ is a multiple of $2abc$, because $2abc$ is a divisor of $6a^2bcx$.

A Common Multiple of two or more algebraic expressions is exactly divisible by each of them.

$12a^3x^3y^3z^3$ is a Common Multiple of $3a^2xy$ and $4a^2yz^2$.

The Lowest Common Multiple (L. C. M.) of two or more algebraic expressions is the expression of lowest degree which is exactly divisible by each of them.

We use the term *Lowest Common Multiple* in algebra because we are concerned about the degree and not about the numerical value.

a^3x is the L. C. M. of a^3 and x; a^4x is also a multiple, and if a is less than 1, a^4x is numerically less than a^3x.

87. Least Common Multiple of Monomials. RULE. *To the Least Common Multiple of the numerical coefficients affix every letter found among the monomials, and to the highest power it occurs in any of them.*

$3\,ax$, $5\,a^2xy$, $4\,ay^4$. By the above rule we write $60\,a^2xy^4$ at once as the L. C. M. of these expressions.

EXERCISES.

Find the L. C. M. of:

1. $2\,ax,\ 3\,a^2x^3y,\ 4\,abxy.$
2. $3\,a^2xy,\ 5\,a^3x^2,\ 15\,a^2x^3y.$
3. $8\,pqr^2,\ 24\,p^2q^2r,\ 12\,pq^3r^2.$
4. $10\,l^2m^3n,\ 15\,l^3mn^3,\ 25\,l^4m^2n.$
5. $7\,ab^3xy,\ 14\,a^3b^2y^4,\ 21\,ab^4x^5y.$
6. $3\,(a-b)\,xy,\ 6\,(a-b)^2x^3y,\ 12\,(a-b)\,x^4y^2.$
7. $4\,(x-y)^3ab,\ 5\,(x-y)^2a^2b^2,\ 10\,(x-y)^2(ab)^3.$
8. $40\,(a^2-x)^3y^2z,\ 60\,(a^2-x)^4yz^2,\ 120\,(a^2-x)^2y^2z^2.$

88. Least Common Multiple of Polynomials. The L. C. M. of polynomials may be found by factoring. Consider each factor as a single quantity and proceed exactly as in the case of monomials.

The L. C. M. of $(x-y)^2$, (x^3-y^3), $x^2-6\,xy+5\,y^2$ is found as follows:

$$(x-y)^2 \equiv (x-y)(x-y),$$
$$(x^3-y^3) \equiv (x-y)(x^2+xy+y^2),$$
$$(x^2-6\,xy+5\,y^2) \equiv (x-y)(x-5\,y).$$

The L. C. M. is $(x-y)^2(x^2+xy+y^2)(x-5\,y)$.

In general the L.C.M. should be left in its factored form; that is, its factors should not be multiplied together.

The L.C.M. is used to a limited extent in reducing fractions to a common denominator.

EXERCISES.

Find the L.C.M. of:

1. a^2-b^2, $a^2-2ab+b^2$, a^2-ab.
2. x^2-16, $x^2-9x-20$.
3. p^2-25, p^2+p-30.
4. l^2-36, $l^2-13l+42$.
5. a^3-4ab^2, a^4-2a^3b.
6. $6a^2$, a^4+3a^2, a^3-3a.
7. m^2+m+1, m^3-1.
8. r^2-5r+6, $r^2+5r-24$.
9. x^3+4x^4, x^4-16x^6.
10. $y^2-9y+14$, $y^2-4y-21$.
11. a^3+8b^3, a^2-4b^2.
12. c^3-27d^3, $c^2-cd-6d^2$.
13. $2-x$, $4-x^2$, $4+x^2$, $16-x^4$.
14. $3+b$, $9-b^2$, $27-b^3$.
15. $x-3$, $x+3$, x^2-6x+9, x^2+6x+9, x^2-9.

CHAPTER VIII.

FRACTIONS.

89. Algebraic Fraction; Numerator; Denominator; Terms. *An algebraic fraction is an indicated division.*

$$a \div b,\ ax^2 \div by,\ (a+b) \div c.$$

It is usual to write these indicated divisions thus:

$$\frac{a}{b},\ \frac{ax^2}{by},\ \frac{a+b}{c},\ \text{or}\ a/b,\ ax^2/by,\ (a+b)/c.$$

$\frac{a}{b}$ is read a over b, the fraction a over b, or a divided by b. The preferred reading is a over b.

The dividend is called the numerator, the divisor the denominator, and the two together the terms of the fraction.

In the fraction $\frac{x}{y}$, x is the numerator, y the denominator, and x and y the terms of the fraction.

Any expression may be put into a fractional form by writing it with a denominator 1; $a \equiv \frac{a}{1}$, $x + y \equiv \frac{x+y}{1}$.

Since a fraction is an indicated division, we know that $\frac{a}{b} \times b = a$; for the fraction $\frac{a}{b}$ may be regarded as the quotient of $a \div b$; but the quotient multiplied by the divisor equals the dividend. Hence, $\frac{a}{b} \times b = a$.

90. The Sign of a Fraction. The sign of a fraction is placed before the line which separates the numerator and denominator. $-\frac{x}{y}$ is read minus the fraction x over y.

Since a fraction is a quotient and the terms are dividend and divisor, the sign of a fraction is determined precisely as the sign of the quotient in division is determined.

$$\frac{+n}{+d}=+\frac{n}{d};\quad \frac{-n}{-d}=+\frac{n}{d};\quad \frac{-n}{+d}=-\frac{n}{d};\quad \frac{+n}{-d}=-\frac{n}{d}.$$

A fraction preceded by a minus sign is equal to the same fraction preceded by a plus sign, provided either the numerator or denominator be preceded by a negative sign.

Thus, $$-\frac{a}{b}=\frac{a}{-b}=\frac{-a}{b}.$$

If the sign of either term of a fraction be changed, the sign of the fraction is changed.

Let $\frac{n}{d}$ be a fraction. Change the sign of n, and it becomes $\frac{-n}{d}=-\frac{n}{d}$; change the sign of d, and it becomes $\frac{n}{-d}=-\frac{n}{d}$.

If the signs of both terms of a fraction are changed, the sign of the fraction is unchanged.

If the signs of n and d are both changed, the fraction $\frac{n}{d}$ becomes $\frac{-n}{-d}=\frac{n}{d}$.

From the above it is evident that the value of a fraction is unaltered by changing the signs of both terms, or by

changing the sign of one term, provided in the latter case the sign of the fraction is also changed.

$$\frac{n}{d}=\frac{-n}{-d};\quad \frac{n}{-d}=-\frac{n}{d};\quad \frac{-n}{d}=-\frac{-n}{-d}=-\frac{n}{d}.$$

The line separating numerator and denominator acts as a vinculum on both terms of the fraction.

$$\frac{x+y}{a+b} \text{ means } (x+y)\div(a+b).$$

$$\frac{a-b}{c}=-\frac{-a+b}{c},\quad \frac{a}{a-x}=-\frac{a}{-a+x}=-\frac{a}{x-a}.$$

91. Law of Signs. If the terms of a fraction are made up of factors, the signs of an even number of factors in either or both terms may be changed without affecting the sign of the fraction. If the signs of an odd number of factors in either term are changed, the sign of that term is changed, and hence the sign of the fraction is also changed.

$$\frac{a(-x)(-y)(z)}{c}=\frac{axyz}{c}.$$

$$\frac{(a-b)(b-c)(c-a)}{(x-y)(y-z)(z-x)}=-\frac{(b-a)(b-c)(c-a)}{(x-y)(y-z)(z-x)}$$
$$=\frac{(b-a)(c-b)(a-c)}{(y-x)(y-z)(z-x)}.$$

92. Reduction of Fractions. *A fraction is reduced when its form is changed without changing its value.*

Reduction of fractions depends upon the following principle:

Multiplying or dividing both terms of a fraction by the same number does not change its value.

$$\frac{n}{d}=\frac{n\times m}{d\times m}.$$

Proof. Let $\frac{n}{d}=f$.

Then $n=fd$, by multiplying both sides by d.

$n\times m=fd\times m$, by multiplying both sides by m.

$\frac{n\times m}{d\times m}=f$, by dividing both sides by $d\times m$.

Hence, $\frac{n}{d}=\frac{n\times m}{d\times m}$, since both $=f$.

Also,
$$\frac{n}{d}=\frac{n\div m}{d\div m}.$$

Proof. Let
$$\frac{n}{d}=f.$$

Then
$$n=fd,$$
$$n\div m=fd\div m,$$
$$\frac{n\div m}{d\div m}=f.$$

Hence,
$$\frac{n}{d}=\frac{n\div m}{d\div m}.$$

Fractions are reduced to *higher* terms by multiplying both terms by the same number; $\frac{x}{y}=\frac{xy^2z}{y^3z}$ (by multiplying both terms by y^2z).

Fractions are reduced to *lower* terms by dividing both terms by the same number; $\frac{a^3x^2y}{a^4x^2y^2}=\frac{y}{ay^2}$ (by dividing both terms by a^3x^2).

A fraction is in its lowest terms when its terms contain no common factors.

To reduce a fraction to its lowest terms, divide or cancel all common factors out of its terms.

(1) $\dfrac{a^2x(a+b)}{a^3(a^2-b^2)}=\dfrac{x}{a(a-b)}$, by dividing out $a^2(a+b)$.

Selecting and canceling the common factors can generally be done mentally.

(2) $\dfrac{x^2-5x+6}{x^2-4}=\dfrac{(x-2)(x-3)}{(x-2)(x+2)}=\dfrac{x-3}{x+2}$.

(3) $\dfrac{(x-a)(x-b)}{(a^2-x^2)(b^2-x^2)}=\dfrac{(a-x)(b-x)}{(a^2-x^2)(b^2-x^2)}=\dfrac{1}{(a+x)(b+x)}$.

EXERCISES.

Cancel factors common to numerator and denominator in the following:

1. $\dfrac{a^4b^2x}{5a^3b^3x^2}$.

2. $\dfrac{3a^2x^2y}{15a^3xy^2}$.

3. $\dfrac{17ab^2x^3}{51b^2cx^2}$.

4. $\dfrac{ax-bx}{abx}$.

5. $\dfrac{ax^2-x^3}{bx^2}$.

6. $\dfrac{ax^n}{bx^{n-1}}$.

7. $\dfrac{a^nb^n}{a^{n-1}b^{n-1}}$.

8. $\dfrac{x^2+ax}{x^2-a^2}$.

9. $\dfrac{x^3-y^3}{x^2-y^2}$.

10. $\dfrac{x^4-x^2y^2}{x^4-y^4}$.

11. $\dfrac{x^2-9}{(x+3)(x-2)}$.

12. $\dfrac{x^3-25x}{(x-5)(x+5)x^2}$.

13. $\dfrac{3x^3-27x}{x^4-6x^2+9}$.

14. $\dfrac{x^2-1}{x^4-1}$.

15. $\dfrac{x^2y-xy^2}{x^4y-xy^4}$.

16. $\dfrac{x^2-y^2}{x^3+y^3}$.

17. $\dfrac{x^3+y^3}{(x+y)^3}$.

18. $\dfrac{(x+y)(x-y)(z-x)}{(x-z)(y-x)(-x-y)}$.

19. $\dfrac{x^2+3x+2}{x^2-1}$.

20. $\dfrac{x^2-5x+6}{x^2+x-12}$.

21. $\dfrac{2x^2+xy-y^2}{(x+y)(x-y)}$.

22. $\dfrac{(x+y)^2-z^2}{(x-y+z)^2}$.

23. $\dfrac{x^2-(x+5)^2}{2x^2+5x}$.

24. $\dfrac{x^2-(a+b)x+ab}{x^2-(b+c)x+bc}$.

25. $\dfrac{x^2-7x+12}{x^2-9x+20}$.

26. $\dfrac{x^2+(a-b)x-ab}{x^2-(b-c)x-bc}$.

27. $\dfrac{x^2+xy+y^2}{x^3-y^3}$.

28. $\dfrac{x^4+6x^3+11x^2+6x}{(x+1)(x^2+5x+6)}$.

29. $\dfrac{x^4-6x^3+11x^2-6x}{(x-3)(x^2-3x+2)}$.

30. $\dfrac{1+a+b+ab}{(1-a^2)(1-b^2)}$.

31. $\dfrac{a^2-b^2+c^2+2ca}{a^2+b^2+2ab-c^2}$.

93. Proper and Improper Fractions. *A proper fraction is one whose numerator is of lower degree in a named letter than the denominator.*

$\dfrac{x^2+2x+5}{x^3+3x^2-7x+4}$ is a proper fraction because its numerator is of second degree in x, while its denominator is of the third degree.

An improper fraction is one whose numerator is of degree equal to or greater than the denominator.

$\dfrac{x^3+3x^2+5x-4}{x^2+2x+1}$ is an improper fraction.

An improper fraction may be reduced to an integral expression and a proper fraction by dividing the numerator by the denominator.

$$\begin{array}{r|l} x^2+2x+1)\; x^3+3x^2+5x-4 & (x+1 \\ \underline{x^3+2x^2+x\quad\;\;} & \\ x^2+4x-4 & \\ \underline{x^2+2x+1} & \\ 2x-5 & \end{array}$$

$$\frac{x^3+3x^2+5x-4}{x^2+2x+1}=x+1+\frac{2x-5}{x^2+2x+1}.$$

The process is similar to the arithmetical process of reducing an improper fraction to a mixed number.

EXERCISES.

Reduce the following to integral or mixed expressions:

1. $\dfrac{x^2+x}{x}$.
2. $\dfrac{x^2-4x+1}{x}$.
3. $\dfrac{3x^2-5x+1}{x+1}$.
4. $\dfrac{x^4-x^3+x+5}{x^3+x^2+x}$.
5. $\dfrac{x^4-4x^3+3x^2-12x}{x-4}$.
6. $\dfrac{x^2+2xy+y^2+z^2}{(x+y)^2}$.
7. $\dfrac{x^3-y^3-3xy(x-y)}{x-y}$.
8. $\dfrac{4x^4-6x^3+12x+5}{x^2+x+1}$.
9. Prove $\dfrac{1}{1-x} \equiv 1+x+x^2+x^3+\dfrac{x^4}{1-x}$.
10. $\dfrac{1}{1+x}$; divide as in Exercise 9 to four terms.
11. $\dfrac{1}{1-x^2}$; " " " " " "
12. $\dfrac{1}{1-3x}$ " " " " " "

94. Reduction of Fractions to a Common Denominator. Since a fraction is not changed in value by multiplying both of its terms by the same number, we may make the denominator any number we please by properly selecting our multiplier. $\dfrac{a}{b}$ and $\dfrac{c}{d}$ may be made to have the common denominator bd.

$$\frac{a}{b}=\frac{a\times d}{b\times d}=\frac{ad}{bd},$$

$$\frac{c}{d}=\frac{c\times b}{d\times b}=\frac{cb}{bd}.$$

The method of reducing fractions to a common denominator is the same as in arithmetic. It may be stated as follows:

RULE. *Find a common denominator, in general the least common denominator* (L. C. D.). *Divide it by the denominator of each fraction, and multiply the terms of the fraction by the quotient.*

(1) Reduce $\frac{a}{b-x}$ and $\frac{c}{b+x}$ to equivalent fractions having the least common denominator.

The L. C. D. is $(b-x)(b+x)$.

$$\text{L. C. D.} \div b - x = b + x,$$

$$\text{L. C. D.} \div b + x = b - x.$$

$$\frac{a}{b-x} = \frac{a(b+x)}{(b-x)(b+x)},$$

$$\frac{c}{b+x} = \frac{c(b-x)}{(b-x)(b+x)}.$$

(2) Reduce $\frac{x}{x^2-4}$, $\frac{4x}{(x-2)(x-3)}$, $\frac{2-x}{(x+2)(x+3)}$ to equivalent fractions having the least common denominator.

The L. C. D. is $(x-2)(x+2)(x-3)(x+3)$.

$$\text{L. C. D.} \div (x^2-4) = x^2-9,$$

$$\text{L. C. D.} \div (x-2)(x-3) = (x+2)(x+3),$$

$$\text{L. C. D.} \div (x+2)(x+3) = (x-2)(x-3).$$

$$\frac{x(x^2-9)}{(x^2-4)(x^2-9)} = \frac{x^3-9x}{(x-2)(x+2)(x-3)(x+3)},$$

$$\frac{4x(x+2)(x+3)}{(x-2)(x-3)(x+2)(x+3)} = \frac{4x^3+20x^2+24x}{(x-2)(x+2)(x-3)(x+3)},$$

$$\frac{(2-x)(x-2)(x-3)}{(x+2)(x+3)(x-2)(x-3)} = \frac{-x^3+7x^2-16x+12}{(x-2)(x+2)(x-3)(x+3)}.$$

EXERCISES.

Reduce to equivalent fractions having lowest common denominators:

1. $\frac{1}{a}, \frac{2}{b}, \frac{2}{3a}, \frac{5}{3b}$, and $\frac{1}{c}$.

2. $\frac{1}{x}, \frac{1}{y}$, and $\frac{1}{x+y}$.

3. $\frac{3}{x+y}$ and $\frac{4}{x-y}$.

4. $\frac{a}{x}, \frac{b}{x+1}$, and $\frac{c}{x+2}$.

5. $\frac{a}{x+1}, \frac{b}{x-1}, \frac{c}{x+2}$, and $\frac{d}{x-2}$.

6. $\frac{x^2+x}{x^2-4}, \frac{3x}{x+2}$, and $\frac{5x}{x-2}$.

7. $\frac{x}{x^2-6x+8}$ and $\frac{3x}{x^2-9x+20}$.

8. $\frac{1}{x^2+5x+6}, \frac{1}{x^2+4x+3}$, and $\frac{1}{x^2+3x+2}$.

9. $\frac{a}{x^2-y^2}, \frac{b}{x^4-y^4}$, and $\frac{c}{x^3-xy^2}$.

10. $\frac{xy}{x^2-y^2}, \frac{y^2}{x^2-xy-2y^2}$, and $\frac{x^2}{x^2+xy-2y^2}$.

11. $\frac{1+x}{1-x}, \frac{3x}{1+x^2}, \frac{5x}{1-x^4}$, and $\frac{1-x}{1+x}$.

12. $\frac{5}{x^2+5x+6}, \frac{6}{x^2+6x+8}$, and $\frac{7}{x^2+7x+12}$.

13. $\frac{1}{x^2-xy+y^2}, \frac{2}{x^2+xy+y^2}$, and $\frac{3}{x^4+x^2y^2+y^4}$.

14. $\frac{a}{b^n c^m}$ and $\frac{3a}{b^{n+1}c^{m+1}}$.

15. $\frac{5}{x^n(a+b)^3}$ and $\frac{6}{x^{n-2}(a+b)^4}$.

95. Addition and Subtraction of Fractions. From division we know that

$$\frac{a+b+c+d-e-f}{g} \equiv \frac{a}{g}+\frac{b}{g}+\frac{c}{g}+\frac{d}{g}-\frac{e}{g}-\frac{f}{g}.$$

If we read this identity from the right, we have the result of adding a number of fractions. Hence, we conclude that

$$\frac{a}{c}+\frac{b}{c}\equiv\frac{a+b}{c}.$$

RULE. *To add or subtract fractions, reduce them to the same denominator and then deal with the numerators according to the rules for addition and subtraction, writing the final result over the common denominator.*

$$\frac{1}{a-x}+\frac{2}{a+x}-\frac{3x}{a^2-x^2}=\text{what?}$$

The common denominator is $(a-x)(a+x)$.

$$\frac{1}{a-x}=\frac{a+x}{a^2-x^2},$$

$$\frac{2}{a+x}=\frac{2a-2x}{a^2-x^2},$$

$$\frac{3x}{a^2-x^2}=\frac{3x}{a^2-x^2}.$$

Thus we have

$$\frac{a+x}{a^2-x^2}+\frac{2a-2x}{a^2-x^2}-\frac{3x}{a^2-x^2}=\frac{a+x+2a-2x-3x}{a^2-x^2}=\frac{3a-4x}{a^2-x^2}.$$

EXERCISES.

Combine and simplify:

1. $x+\dfrac{1}{x-1}.$

2. $3x-\dfrac{3}{x+1}.$

3. $5x-\dfrac{5x^2}{x+10}.$

4. $x+1-\dfrac{x^2+3x}{x+2}.$

5. $x^2+2x+5+\dfrac{10}{x-2}.$

6. $x+a+\dfrac{x-a}{(x+a)^2}.$

7. $\dfrac{5}{x}+\dfrac{3}{x}.$

8. $\dfrac{7x}{x+a}+\dfrac{5x}{x+a}-\dfrac{3x}{x+a}+\dfrac{1}{x+a}.$

9. $\dfrac{3x^2+x}{x^3+a^3}+\dfrac{5x^2-3x+4}{x^3+a^3}-\dfrac{6x^2+4x+1}{x^3+a^3}$.

10. $\dfrac{1}{x+1}+\dfrac{1}{x-1}$.

11. $\dfrac{2}{x^2+2}+\dfrac{5}{x^2-2}$.

12. $\dfrac{3x+1}{x^2-1}-\dfrac{5x-2}{(x+1)^2}+\dfrac{1}{x-1}$.

13. $\dfrac{a}{a+b}-\dfrac{b}{a-b}-\dfrac{3ab}{a^2-b^2}$.

14. $\dfrac{3x+5}{x^2-9}-\dfrac{5x+2}{x^2-5x+6}$.

15. $\dfrac{2x+1}{x^2+2x+1}+\dfrac{6x}{x^2-1}-\dfrac{8x+1}{(x-1)^2}$.

16. $\dfrac{5}{(x-1)(x-2)(x+4)}-\dfrac{3}{(x-1)(x+4)}$.

17. $\dfrac{2x}{(x+3)(x-2)(x+1)}+\dfrac{x+2}{(x+4)(x^2-x-2)}$.

18. $\dfrac{1}{(x+y)^2-z^2}-\dfrac{1}{x^2-(y+z)^2}$.

19. $\dfrac{1}{x^2-5x+6}-\dfrac{1}{x^2-6x+8}$.

20. $\dfrac{x}{(x-y)(y-z)}+\dfrac{y}{(y-z)(z-x)}+\dfrac{z}{(z-x)(x-y)}$.

96. Multiplication of Fractions.

To prove $\dfrac{a}{b}\times\dfrac{c}{d}=\dfrac{ac}{bd}$.

Let $\dfrac{a}{b}=f_1,$ (1)

and $\dfrac{c}{d}=f_2.$ (2)

Then $\dfrac{a}{b}\times\dfrac{c}{d}=f_1f_2.$

But from (1) $a=f_1b,$ (3)

and from (2) $c=f_2d.$ (4)

Now multiplying Equation (3) by (4), member by member,

$$ac = f_1 f_2 bd.$$

$$\frac{ac}{bd} = f_1 f_2, \text{ by dividing both sides by } bd.$$

But $$\frac{a}{b} \times \frac{c}{d} = f_1 f_2.$$

Hence, $$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

RULE. *The product of two fractions is the product of the numerators over the product of the denominators.*

This covers all possible cases of multiplication involving fractions, for all integers can be put in fractional form.

$$\frac{a}{b} \times c = \frac{a}{b} \times \frac{c}{1} = \frac{ac}{b}.$$

$$a \times \frac{c}{d} = \frac{a}{1} \times \frac{c}{d} = \frac{ac}{d}.$$

Any number of fractions are multiplied together by placing the product of all the numerators over the product of all the denominators.

In all problems in multiplication of fractions free use of cancellation should be made.

$$\frac{x+y}{a^2-b^2} \times \frac{a-b}{x-y} \times \frac{a+b}{x+y} \times \frac{x-y}{x^2+1}.$$

In this, if we cancel the common factors, our work appears thus:

$$\frac{\cancel{x+y}}{(\cancel{a+b})(\cancel{a-b})} \times \frac{\cancel{a-b}}{\cancel{x-y}} \times \frac{\cancel{a+b}}{\cancel{x+y}} \times \frac{\cancel{x-y}}{x^2+1} = \frac{1}{x^2+1}.$$

EXERCISES.

Perform the operations indicated, reducing the fractions to lowest terms:

1. $\frac{3x}{5y} \times \frac{10y}{12x^2}$.

2. $\frac{5x^2}{3y^2} \times \frac{18y^3}{15x^3}$.

3. $\frac{3(x+y)}{2(x-y)} \times \frac{6(x-y)^2}{9(x+y)^2}$.

4. $\frac{a(x^2-y^2)}{b(x+y)^2} \times \frac{b^2}{a^3} \times \frac{x+y}{x-y}$.

5. $\frac{12ax^2y}{16bxy^2} \times \frac{14b^2y^3}{6a^2x^3} \times \frac{y^2(x^3+y^3)}{x(x+y)}$.

6. $\frac{5a(x+y)^2}{6b(x-y)^2} \times \frac{2}{3}\left(\frac{x-y}{x+y}\right)^2$.

7. $\left(\frac{a}{x}-1\right) \times \left(\frac{a}{x}+1\right) \equiv \frac{a^2}{x^2}-1$.

8. $\left(\frac{x}{y}-z\right)\left(\frac{x}{y}+z\right)$.

9. $\left(\frac{a}{b}+\frac{c}{d}\right)^2$.

10. $\left(\frac{a}{b}-\frac{c}{d}\right)\left(\frac{a}{b}+\frac{c}{d}\right)$.

11. $\left(\frac{a}{b}\right)^{10} \times \left(\frac{b}{c}\right)^{12} \times \left(\frac{c}{d}\right)^{14} \times \left(\frac{d}{a}\right)^{16}$.

12. $\left(\frac{x}{y}+\frac{y}{x}\right)\left(\frac{x}{y}+\frac{y}{x}\right)$.

13. $\left[\left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right)\left(\frac{c}{d}\right) + \left(\frac{c}{d}\right)^2\right]\left(\frac{a}{b}+\frac{c}{d}\right)$.

14. $\left(\frac{x}{y}-\frac{y}{x}\right)^2$.

15. $\left(\frac{1}{x^2}+\frac{1}{x}+1\right)\left(\frac{1}{x}-1\right)$.

16. $\frac{(x+3)(x+4)}{x^2+5x+6} \times \frac{x^2+4x+3}{(x+1)(x+4)}$. (Cancel common factors.)

17. $\frac{x^2+7x+12}{x^2+4x+3} \times \frac{x^2+6x+5}{x^2+9x+20}$.

18. $\frac{x^2-9x+20}{x^2-7x+12} \times \frac{x^2-4x+3}{x^2-6x+5}$.

19. $\left(\frac{x+1}{x+5}\right)^2 \times \left(\frac{x^2-25}{x^2-1}\right) \times \left(\frac{x-1}{x-5}\right)$.

20. $\frac{x^{2n}-4x^n+4}{y^{2m}+2y^m+1} \times \frac{y^{2m}-1}{x^{2n}-4}$.

97. Division of Fractions.

To prove $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$

Let $$\frac{a}{b} = f_1, \qquad (1)$$

$$\frac{c}{d} = f_2. \qquad (2)$$

Then $\frac{a}{b} \div \frac{c}{d} = \frac{f_1}{f_2} = f_1 \div f_2.$

But from (1) $a = f_1 b,$

and from (2) $c = f_2 d.$

Then $\frac{a}{c} = \frac{f_1 b}{f_2 d} = \frac{f_1}{f_2} \times \frac{b}{d}.$

$\frac{a}{c} \times \frac{d}{b} = \frac{f_1}{f_2},$ by multiplying both sides by $\frac{d}{b}.$

$$\frac{a}{c} \times \frac{d}{b} = \frac{ad}{cb} = \frac{a}{b} \times \frac{d}{c} = \frac{f_1}{f_2}.$$

Hence, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$

RULE. *To divide one fraction by another, invert the divisor and multiply.*

Since all integers can be expressed in fractional form, this rule suffices for all forms of division involving the fraction.

(1) $$\frac{x}{y} \div a = \frac{x}{y} \div \frac{a}{1} = \frac{x}{y} \times \frac{1}{a} = \frac{x}{ay}.$$

This is dividing a fraction by an integer.

(2) $$a \div \frac{x}{y} = \frac{a}{1} \div \frac{x}{y} = \frac{a}{1} \times \frac{y}{x} = \frac{ay}{x}.$$

This is dividing an integer by a fraction.

(3) $\dfrac{a^2-b^2}{x+y} \div \dfrac{a-b}{x^2-y^2} = \dfrac{a^2-b^2}{x+y} \times \dfrac{x^2-y^2}{a-b} = (a+b)(x-y).$

In this case the common factors $(a-b)$ and $(x+y)$ are canceled. The student should constantly be on the alert for common factors and cancel them as soon as they appear.

EXERCISES.

1. $\dfrac{3\,ax}{5\,a^2y} \div \dfrac{6\,a^2x}{10\,a^3y^2}.$

2. $\dfrac{4\,a^2x^2}{15\,b^2y^2} \div \dfrac{8\,ax}{3\,by}.$

3. $\dfrac{12\,x^2y^2z^2}{9\,x^3yz^3} \div \dfrac{4\,axy^2}{3\,x^2yz}.$

4. $\dfrac{x+y}{(x-y)^2} \div \dfrac{(x+y)^2}{x-y}.$

5. $\dfrac{a+3\,b}{a^2+5\,b} \div \dfrac{ab+3\,b^2}{a^3+5\,ab}.$

6. $\dfrac{x-4\,y}{3\,x^2-5\,xy} \div \dfrac{3\,x^3-5\,x^2y}{2\,x^2-8\,xy}.$

7. $\dfrac{x+y}{x^2-y^2} \div \dfrac{(x+y)^2}{(x-y)^2}.$

8. $\dfrac{x^2-y^2}{x^3-y^3} \div \dfrac{x+y}{x^2+xy+y^2}.$

9. $\dfrac{x^2-9\,y^2}{(x+3\,y)^2} \div \dfrac{x-3\,y}{x+3\,y}.$

10. $\dfrac{(x+a)(x-b)}{(x-a)(x+b)} \div \dfrac{x^2+(a-b)x-ab}{x^2-(a-b)x-ab}.$

11. $\dfrac{x^2-y^2}{x^4-y^4} \div \dfrac{(x-y)}{(x^2+y^2)}.$

12. $\dfrac{a^2-(b+c)^2}{b^2-(c+a)^2} \div \dfrac{a-b-c}{-a+b-c}.$

98. The Complex Fraction. *A fraction which has a fractional expression for either or both of its terms is called a complex fraction.*

$$\frac{\dfrac{a}{b}}{\dfrac{c}{d}},\quad \frac{\dfrac{1}{x}-\dfrac{1}{y}}{\dfrac{1}{x}+\dfrac{1}{y}},\quad \frac{x+\dfrac{1}{y}}{\dfrac{a+x}{y}}$$ are complex fractions.

Since a fraction is an indicated division, a complex fraction is simplified by performing the division indicated.

(1) $$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}.$$

(2) $$\frac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{x}+\frac{1}{y}}=\frac{\frac{y}{xy}-\frac{x}{xy}}{\frac{y}{xy}+\frac{x}{xy}}=\frac{\frac{y-x}{xy}}{\frac{y+x}{xy}}=\frac{y-x}{xy}\div\frac{y+x}{xy}$$

$$=\frac{y-x}{xy}\times\frac{xy}{y+x}=\frac{y-x}{y+x}.$$

EXERCISES.

Simplify the following:

1. $\dfrac{\frac{x}{y}+1}{\frac{y}{x}+1}$.

2. $\dfrac{1+\frac{a}{x-a}}{2-\frac{2a}{x+a}}$.

3. $\dfrac{x}{x+\frac{1}{x}}$.

4. $\dfrac{x+y}{1-\frac{x}{x+y}}$.

5. $\dfrac{x}{1+\frac{x}{y}}\div\dfrac{y}{1+\frac{y}{x}}$.

6. $\dfrac{x+\frac{6}{x}-5}{1+\frac{8}{x^2}-\frac{6}{x}}$.

7. $\dfrac{\frac{x+a}{x-a}-\frac{x-a}{x+a}}{\frac{x+a}{x-a}+\frac{x-a}{x+a}}$.

8. $\dfrac{\frac{x}{y^2}-\frac{y}{x^2}}{\frac{1}{x^2}+\frac{1}{xy}+\frac{1}{y^2}}$.

9. $\dfrac{b-2-\frac{6}{b+3}}{b-4+\frac{6}{b+3}}$.

10. $\dfrac{\frac{1}{x^2+3\,xy+2\,y^2}}{\frac{x}{x+y}+\frac{2\,y}{x+2\,y}}$.

11. $3+\dfrac{1}{3-\frac{1}{x}}$.

12. $2-\dfrac{1}{2-\frac{1}{x}}$.

CHAPTER IX.

EQUATIONS IN ONE VARIABLE.

99. Identity; Conditional Equation. Distinctions between an *identity* and a *conditional equation* have already been made in Chapter II. We have had illustrations of the identity in all the fundamental operations and in factoring.

$$(x+2)x+4 \equiv x^2+2x+4,$$

and
$$(x+1)(x-5) \equiv x^2-4x-5$$

are identities; that is, they are equalities that are true for all values of x.

A *conditional equation* restricts the value of some *one* letter, the letter so restricted being called the *variable.* The other numbers of an equation are called *constants.*

(1) $x=5$ restricts x to the value 5.

(2) $3x=18$ restricts x to the value 6.

(3) $ax=b$ restricts x to the value $\frac{b}{a}$.

(4) $x^2=4x$ restricts x to the values 0 and 4.

Each of these equations becomes an identity for the restricted value of the variable.

Thus, (1) becomes $5 \equiv 5$ for $x=5$,

(2) becomes $3 \times 6 \equiv 18$ for $x=6$,

(3) becomes $a \times \frac{b}{a} \equiv b$ for $x = \frac{b}{a}$,

(4) becomes $0^2 \equiv 4 \times 0$ for $x = 0$,

and $(4)^2 \equiv 4 \times 4$ for $x = 4$.

100. Root of an Equation. *A value of the variable for which the equation becomes an identity, is called a root of the equation.*

A root of an equation is frequently called a *solution.*

Any root of an equation when substituted for the variable is said to *satisfy* the equation; that is, an equation is *satisfied* by any value of the variable which reduces it to an identity.

$2x + 3 = 15$ has 6 for a root. Substituting 6 for x, this equation becomes $2 \times 6 + 3 \equiv 15$, which is an identity. 6 is the value of x which satisfies the equation $2x + 3 = 15$.

101. Classes of Equations. *A rational equation is one in which the variable is free from radical signs.*

$3x^2 - 7x + 11 = 0$ is a rational equation.

An irrational equation is one in which the variable is affected by a radical sign.

$5x^2 - 6\sqrt{x} = 12$ is an irrational equation.

An integral equation is one in which the variable appears only in the numerators of its terms.

$3x^2 + \frac{5x}{17} + \frac{9}{13} = \frac{7}{11}$ is an integral equation.

A fractional equation is one in which the variable appears in one or more denominators.

$3x + \frac{7}{x} = \frac{3 - x}{x^2 - 4x + 1}$ is a fractional equation.

A linear equation is one in which the variable appears to the first degree only.

$3x + 7 = 11$ is a linear equation.

A quadratic equation is one in which the highest power of the variable is two.

$5x^2 + 7x - 16 = 0$ is a quadratic equation.

A cubic equation is one in which the highest power of the variable is three.

$x^2 - 6x^2 + 11x - 6 = 0$ is a cubic equation.

It should be noticed that several of these terms may be applied to the same equation.

$3x^2 - 7x + 13 = 0$ is a rational, integral, and quadratic equation.

EXERCISES.

Classify the following equations:

1. $ax^2 + bx + c = 0.$

2. $ax + b = 0.$

3. $ax^3 + bx^2 + cx + d = 0.$

4. $x^2 + \frac{3}{4}x + \frac{7}{8} = 0.$

5. $x^3 + \frac{5}{6}x^2 - \frac{7}{8}x = \frac{13}{14}.$

6. $3\sqrt{x} = 11 - x.$

7. $\frac{3}{x} - \frac{5 + 2x}{x^2 + 1} = 10.$

102. Equivalent Equations. It will be noticed that the equations

$$(1)\quad x - 4 = 0 \text{ and}$$

$$(2)\quad 3x - 12 = 0$$

have the same root, viz., $x = 4$.

The equations

$$(3)\quad 5x^2 - 10x = 0 \text{ and}$$

$$(4)\quad 5x^2 - 10x + 6 = 6$$

are satisfied when $x = 0$ and when $x = 2$. Such equations as (1), (2), and (3), (4), having the same roots, are called *equivalent equations.*

Equivalent equations are those having precisely the same roots.

Determine which of the following equations are equivalent:

(1) $x - 1 = 0$, whose root is 1.

(2) $3x - 6 = 0$, whose root is 2.

(3) $2x - 2 = 0$, whose root is 1.

(4) $x^2 - x = 0$, whose roots are 0 and 1.

(5) $2x - 4 = 0$, whose root is 2.

(6) $x^2 - 1 = 0$, whose roots are 1 and -1.

(7) $3x - 6 + 5 = 5$, whose root is 2.

(8) $x^2 - x - 4 = -4$, whose roots are 0 and 1.

(9) $\frac{x}{4} - \frac{1}{2} = 0$, whose root is 2.

(10) $\frac{x^2}{2} - \frac{x}{2} + 1 = 1$, whose roots are 0 and 1.

(11) $x^2 + 3 = 4$, whose roots are 1 and -1.

103. Solution of Linear Equations. *To solve an equation is to find all of its roots.*

The solution of an equation consists in deriving one or more equivalent equations, the last of which is the value of the variable.

Thus, to solve

(1) $$3x + 4 - 2 = 6 + 2x,$$

we bring $2x$ to the first member of the equation and $(4-2)$ to the second member. This is done by subtracting these quantities from both members and gives us

(2) $$3x - 2x + 4 - 2 - (4 - 2) = 6 + 2x - 2x - (4 - 2),$$

which becomes, by omitting the terms that destroy each other,

(3) $$3x - 2x = 6 - 4 + 2.$$

This process is called *transposition*. It is effected by changing the sign of a term when it is moved from one member of an equation to the other.

We next unite the terms in each member of Equation (3), which makes

(4) $$x = 4.$$

This process is called *combining terms*. It is effected by addition. The last equation expresses the value of the variable x. Hence, 4 is the *root* or *solution* of the given equation.

It should be noticed that Equations (1), (2), (3), and (4) are equivalent equations, each having the root 4.

If 4 is put for x in these equations, they become

(1) $$3 \times 4 + 4 - 2 = 6 + 2 \times 4,$$

(2) $$3 \times 4 - 2 \times 4 + 4 - 2 - (4 - 2)$$
$$= 6 + 2 \times 4 - 2 \times 4 - (4 - 2),$$

(3) $$3 \times 4 - 2 \times 4 = 6 - 4 + 2,$$

(4) $$4 = 4.$$

Each of the above is an identity.

As another illustration let us solve

(1) $$4x + 2 - 7 = 2x - 8.$$

Transposing 2 and -7 to the second member and $2x$ to the first member, we have

(2) $$4x - 2x = -2 + 7 - 8.$$

Combining, we have

(3) $$2x = -3.$$

Dividing both terms of the equation by 2, we have

(4) $$x = -\tfrac{3}{2}.$$

The root or solution of Equation (1) is $-\frac{3}{2}$. Equations (1), (2), (3), and (4) are equivalent equations, each being satisfied by the root $-\frac{3}{2}$.

Putting $x = -\frac{3}{2}$, they become

(1) $$4(-\tfrac{3}{2}) + 2 - 7 = 2(-\tfrac{3}{2}) - 8,$$

or $$-6 + 2 - 7 = -3 - 8.$$

(2) $$4(-\tfrac{3}{2}) - 2(-\tfrac{3}{2}) = -2 + 7 - 8,$$

or $$-6 + 3 = -2 + 7 - 8.$$

(3) $$2(-\tfrac{3}{2}) = -3.$$

(4) $$-\tfrac{3}{2} = -\tfrac{3}{2}.$$

As a further illustration let us solve

(1) $$\frac{3x - 5}{2} + 6 = \frac{x}{5} - \frac{2x - 1}{3} + 13.$$

In order to get rid of the fractions in this equation, we multiply both members of the equation by the Lowest Common Multiple of the denominators. Multiply both sides by 30, the L. C. M. of 2, 5, and 3; the result is

(2) $$45x - 75 + 180 = 6x - 20x + 10 + 390.$$

This process is called *clearing of fractions.* It is always brought about by multiplying both members of the equation by the L. C. M. of the denominators.

(3) $45x - 6x + 20x = 75 - 180 + 10 + 390$, by transposing.

(4) $59x = 295$, by combining.

(5) $x = 5$, by dividing.

5 is the *root* or solution of (1).

Equations (1), (2), (3), (4), and (5) are *equivalent equations.*

Show that each is satisfied by the root 5.

104. Rule for Solving a Linear Equation.

(1) *Clear of fractions.*

(2) *By transposition bring all the terms containing the variable to one member of the equation and all the constant terms to the other member.*

(3) *Combine the terms of each member of the equation by addition.*

(4) *Divide both members of the equation by the coefficient of the variable.*

Steps (1) and (4) depend upon the axiom that multiplying or dividing equals by equals gives equals. Steps (2) and (3) depend upon the axiom that increasing or diminishing equals by equals gives equals.

EXERCISES.

Solve the following equations:

1. $3x - 5 = 19$.
2. $7x - 11 = 24$.
3. $3x - 5 = x + 13$.
4. $7x - 12 + 3 = 5x + 16 - 5$.
5. $12 - 4x + 2 = 13 - 7x + 10$.
6. $15 - 14x - 7 = 17 - 16x - 6$.

7. $7x-11+4x-7=3x-8.$

8. $11-5x+17-3x=18-11x+23.$

9. $5x-16-6x-6=115-7x-4x-7.$

10. $2x-22+7x+14=6x-8+4x+42-5x.$

11. $10(x-3)=8(x-2).$ (Remove the parentheses first.)

12. $11(4x-5)=7(6x-5).$

13. $3(x-2)+2(2x-3)=3(x-4).$

14. $\frac{3x+1}{2}=\frac{x+12}{3}.$

15. $\frac{5x-4}{3}=\frac{2x+3}{2}.$

16. $\frac{7x-3}{2}+1=\frac{2x+5}{3}.$

17. $\frac{8x+5}{5}-2=\frac{3x+4}{5}.$

18. $\frac{x}{2}+\frac{x}{3}=6-\frac{2x}{3}-1.$

19. $\frac{x}{2}-\frac{x}{3}=\frac{x}{4}+\frac{1}{2}.$

20. $\frac{2x-1}{3}+\frac{3x-5}{2}=\frac{6x+2}{4}.$

21. $x-\frac{3x-3}{5}+\frac{x}{2}=4+\frac{x}{6}.$

22. $2x-\frac{4x-2}{5}+\frac{x}{3}=\frac{2x}{3}+3.$

23. $\frac{x+1}{8}-\frac{2x-5}{9}+\frac{x}{7}=1.$

24. $\frac{x-1}{6}+\frac{2x-6}{5}=1+\frac{x-3}{2}.$

25. $\frac{x+1}{9}-\frac{2x+1}{7}+4=\frac{x-9}{8}.$

26. $\frac{2x-1}{3}+3x-\frac{5x+3}{7}=12+\frac{2x}{5}.$

27. $3(x-2)-2(x-5)+2x-20=17.$

28. $ax+bx=a^2+2ab+b^2.$

$$(a+b)x=(a^2+2ab+b^2).$$

$$x=\frac{a^2+2ab+b^2}{a+b},$$

$$x=a+b.$$

29. $ax + a^2 = bx + b^2$.

30. $ax - bx = a^3 - b^3$.

31. $ax + bx = a^3 + b^3$.

32. $a^2x - abx + b^2x = a^3 + b^3$.

33. $a^2x + b^3 = a^3 - b^2x - abx$.

34. $cx - 3x = c^2 - 9$.

35. $mx - 5nx = 3m^2 - 75n^2$.

36. $\frac{x}{a} - \frac{x}{b} = b^2 - a^2$.

37. $\frac{x}{a+b} - \frac{x}{a-b} = 2b$.

38. $(x-3)(x+5) - 7 = (x+4)(x-8)$.

39. $(2x-5)^2 + 4 = (x-6)(4x-3)$.

40. $14 - (2-x)^2 = 5 - (x+3)(x-2)$.

EXERCISES.

1. One half of A's money is \$35 more than B's. They together have \$280. How much has each?

Solution.

Let

$$\begin{aligned} x &= \text{B's money.} \\ x + \$35 &= \text{one half of A's money.} \\ 2x + 70 &= \text{A's money.} \\ x + 2x + 70 &= 280. \\ x + 2x &= 280 - 70. \\ 3x &= 210. \\ x &= 70 = \text{B's money.} \\ 2x + 70 &= 210 = \text{A's money.} \end{aligned}$$

Verification. One half of \$210 is \$105, which is \$35 more than \$70. Also, \$70 + \$210 = \$280.

2. A has \$10 more than 3 times as much as B, and they together have \$250. How much has each?

3. Find two numbers whose sum is 81, such that one may exceed 6 times the other by 4.

4. Divide 114 into three parts such that the first may exceed the second by 15, and the third the first by 21.

5. Divide \$176 among A, B, and C, so that B may have \$16 less than A, and \$8 more than C.

6. Divide 440 into three parts such that the second is double the first increased by 10, and the third is the sum of the first and second.

7. What two numbers have a sum of 861 and a difference of 221?

8. Find a number that exceeds 31 by the same amount that $\frac{1}{6}$ of the number exceeds 1.

SOLUTION.

Let $x =$ the number.

$x - 31 =$ the excess of the number over 31.

$\frac{x}{6} - 1 =$ the excess of $\frac{1}{6}$ of the number over 1.

By the conditions of the problem, these are the same; hence,

$$x - 31 = \frac{x}{6} - 1.$$

$$6x - 186 = x - 6.$$

$$6x - x = 186 - 6.$$

$$5x = 180.$$

$$x = 36.$$

VERIFICATION. $36 - 31 = 5.$

$\frac{1}{6}$ of $36 - 1 = 5.$

9. What number increased by $\frac{1}{2}$ of itself and 80 is 30 more than double itself?

10. Eight times the difference between the third and fourth parts of a certain number is 40 less than the number. What is the number?

11. If 10 be subtracted from a number, $\frac{1}{2}$ the remainder $+ 40$ is 30 less than the number. What is the number?

12. Find two consecutive numbers such that $\frac{1}{3}$ of one plus $\frac{1}{4}$ of the other is 44.

SUGGESTION. Let $x =$ one number, and $x + 1 =$ the other number.

13. Find two consecutive numbers such that $\frac{1}{3}$ their sum is 34 less than the larger one.

14. Find three consecutive numbers such that $\frac{1}{4}$ the first $+ \frac{1}{3}$ the second $+ \frac{1}{2}$ the third is 88.

15. In 10 years John will be twice as old as Henry was 10 years ago. John is 9 years older than Henry. Find their ages now.

SOLUTION.

Let

$$
\begin{aligned}
x &= \text{Henry's age.}\\
x + 9 &= \text{John's age.}\\
x + 9 + 10 &= \text{John's age 10 years hence.}\\
x - 10 &= \text{Henry's age 10 years ago.}\\
x + 9 + 10 &= 2\,(x - 10).\\
x + 9 + 10 &= 2\,x - 20.\\
x - 2\,x &= -\,9 - 10 - 20.\\
-\,x &= -\,39.\\
x &= 39.\\
x + 9 &= 48.
\end{aligned}
$$

16. A man's age plus that of his wife's is 95 years; 40 years ago he was twice as old as she was then. What are their ages now?

17. Eight years ago a father was 9 times as old as his son was at that time; in 37 years the father will be $1\frac{1}{2}$ times as old as the son is at that time. What are their ages now?

18. A man left $\frac{1}{3}$ his estate to his son, $\frac{1}{7}$ to a nephew, $\frac{1}{9}$ to a niece, and the remainder, amounting to \$2600, to his wife. What was the value of his estate?

19. A house is sold for $2280. This is a gain of 14%. What did the house cost?

SOLUTION.

Let $x =$ the cost of the house.

$$\tfrac{14}{100}x = \text{gain}.$$

$$x + \tfrac{14}{100}x = 2280.$$

$$100x + 14x = 228000.$$

$$114x = 228000.$$

$$x = 2000.$$

20. A horse sold at a loss of 7% brought $111.60. What did the horse cost?

21. A man invests $\frac{1}{3}$ his capital at 4% and the remainder at 5%. His income is $2800. What is his capital?

22. What number must be added to each of the terms of the fraction $\frac{11}{13}$ to make it $\frac{24}{25}$?

23. What number must be subtracted from both terms of the fraction $\frac{19}{37}$ to make it $\frac{2}{5}$?

24. Divide $5600 into two parts such that the income from one part at 3% may be equal to the income of the other part at 4%.

25. Divide $760 among A, B, C, and D so that A and B together shall receive $150, A and C together $190, and A and D together, $580.

26. $7.20 is changed into 36 coins. Each coin is either a dime or a quarter. How many of each are there?

27. A bill of $10.20 is paid in an equal number of dimes, quarters, and half dollars. How many of each are used?

28. A man bought sheep at $4 a head, calves at $9, and cows at $35. He bought twice as many calves as cows, and twice as many sheep as calves. The cost of all the stock was $690. How many head of each did he buy?

29. Find three consecutive numbers such that the sum of the quotient of the first divided by 10, the second by 11, and the third by 61, is 25.

30. Find three numbers such that the second is a times the first, the third b times the second, and their sum c.

31. One half of A's money is equal to B's, and five eighths of B's is equal to C's; together they have $1450. How much has each?

32. A man walks out at the rate of 4 miles an hour, and rides back at the rate of 10 miles an hour. How far can he go out if he must make the round trip in 7 hours?

33. A man sold 12 acres more than $\frac{1}{4}$ of his farm, and had 2 acres less than $\frac{5}{8}$ of it left. How many acres had he?

34. A train leaves a station at 8 A.M. and runs 30 miles an hour. At 11 A.M. another train leaves in the same direction running 45 miles an hour. When and where will it overtake the first train?

35. A and B are two towns 120 miles apart. A messenger starts from A to B at 7 A.M. and travels 10 miles an hour. At 8 A.M. another messenger starts from B to A and travels 12 miles an hour. When and where will they meet?

36. A man in traveling from New York to Buffalo, goes $\frac{1}{2}$ as far by boat as by train and $\frac{1}{16}$ as far by carriage as by boat. If the distance to Buffalo from New York be 490 miles, how far does he travel in each conveyance?

105. The Linear Type. Every linear equation in a single variable may be reduced to the type form

$$ax + b = 0.$$

In this form a and b represent any positive or negative numbers whatever.

For example, $3x - 7 + \frac{5x - 1}{2} = 2x - 7 + \frac{3x}{4}$.

Clearing of fractions,

$$12x - 28 + 10x - 2 = 8x - 28 + 3x.$$

Transposing all terms to the first member,

$$12x + 10x - 8x - 3x - 28 - 2 + 28 = 0.$$

Collecting, $11x - 2 = 0$.

This is in the type form.

Comparing it with $ax + b = 0$, we see that $a = 11$ and $b = -2$.

The solution of the type form $ax + b = 0$

is
$$x = -\frac{b}{a}.$$

Hence, the solution of the above example is $x = \frac{2}{11}$.

Special roots of $ax + b = 0$.

If $b = 0$, then the solution of $ax + b = 0$ becomes

$$x = -\frac{0}{a} = 0.$$

If $a = 0$ and b is not 0, then the solution of $ax + b = 0$ becomes

$$x = -\frac{b}{0}.$$

We have here a new form whose value we must investigate.

$$\frac{+b}{1} = +b,$$

$$\frac{+b}{.1} = +10b,$$

$$\frac{+b}{.01} = +100b,$$

$$\frac{+b}{.001} = +1000b,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{+b}{.00000001} = +100000000b.$$

It appears that, as we decrease the denominator, the value of the fraction increases. When the denominator of the fraction is very small, the value of the fraction is very large. When the denominator becomes 0, the value of the fraction is large beyond measure. We express this fact by saying that the value of the fraction is *infinity*. The symbol for infinity is ∞.

$$x = -\frac{b}{0} = -\infty.$$

Any number divided by 0 is equal to ∞.

$$\frac{3}{0} = \infty,\ \frac{15}{0} = \infty,\ \frac{a}{0} = \infty,\ \frac{1000\,b}{0} = \infty, \text{ etc.}$$

If $a = 0$ and $b = 0$, the solution of $ax + b = 0$ becomes

$$x = -\frac{0}{0}.$$

$\frac{0}{0}$ is the symbol of indeterminateness. $\frac{0}{0}$ may have any value.

(1) $$\frac{x^2 - 1}{x - 1} = x + 1.$$

If in the above we put $x = 1$, it becomes

$$\frac{1 - 1}{1 - 1} = 1 + 1, \text{ or } \frac{0}{0} = 2.$$

(2) $$\frac{x^2 - 25}{x - 5} = x + 5.$$

If in this we put $x = 5$, it becomes

$$\frac{25 - 25}{5 - 5} = 5 + 5, \text{ or } \frac{0}{0} = 10.$$

106. Equations of Second or Higher Degree which depend upon the Linear Type.

If we have the equation

$$x^2 - 5x + 6 = 0,$$

we may by factoring write it in the form

$$(x - 3)(x - 2) = 0.$$

We know that if *one* factor of a product is 0, the product is 0. The product $(x - 3)(x - 2)$ may be 0 by either factor being 0. If $x - 3 = 0$, then the product is 0, or if $x - 2 = 0$, then the product is 0; that is, the product is 0 if $x = 3$, or $x = 2$.

This is called equating the factors to 0.

A *root* is a value of the variable which satisfies the equation. Hence, in the above equation, 3 is a root because it satisfies the equation. 2 is likewise a root because it also satisfies the equation. Therefore, the equation $x^2 - 5x + 6 = 0$ has the two roots $x = 3$ and $x = 2$.

An equation of higher degree than the first may be solved by the linear type, provided, after all the terms have been brought to one member, it may be factored into linear factors. Each factor equated to 0 will give one root.

Hence, *the number of roots is equal to the degree of the equation.*

EXERCISES.

1. $x^2 - 5x - 24 = 0.$

By factoring, this is written $(x - 8)(x + 3) = 0.$

Hence, $\quad x - 8 = 0$ or $x = 8$,

also $\quad x + 3 = 0$ or $x = -3$.

The two roots are 8 and -3.

2. $x^2 - 3x - 40 = 0.$
3. $x^2 + 6x + 8 = 0.$
4. $x^2 + 10x + 16 = 0.$
5. $x^2 - 5x - 14 = 0.$
6. $x^2 - 16x + 48 = 0.$
7. $x^2 + 4 = 4x.$
8. $x^2 - 1 = 0.$
9. $x^2 - 25 = 0.$
10. $x^2 + 11 = 36.$
11. $x^2 - 16 = 0.$
12. $x^2 - (a - b)^2 = 0.$
13. $16x^2 - 25 = 0.$
14. $x^2 - 7x + 10 = 0.$
15. $x^2 + 3x - 10 = 0.$
16. $x^2 + 8x + 15 = 0.$
17. $4x^2 - 12x + 9 = 0.$
18. $x^2 - b^2 + 2ax + a^2 = 0.$
19. $x^2 - 12x + 35 = 0.$
20. $x^2 - 21x + 20 = 0.$
21. $x^2 + 28x + 75 = 0.$
22. $x^2 - 7x = 98.$
23. $x^2 + 7x - 98 = 0.$
24. $3x^2 + 11x - 4 = 0.$
25. $12(x + 1) - 3(x - 1) + x^2 - 1 = 0.$
26. $(x^2 - x)^2 - 22(x^2 - x) + 40 = 0.$
27. $(x^2 + 3x)^2 - 8(x^2 + 3x) - 20 = 0.$
28. $x^3 + 5x^2 + 6x = 0.$
29. $x^3 - 12x^2 + 27x = 0.$
30. $(x+4)^2 + (2x-5)^2 = 73.$
31. $(2x-5)^2 - (2x+10)^2 = 24.$
32. $(5x + 4)^2 - (3x-8)^2 = 0.$
33. $y^2 - \frac{9}{25} = 0.$
34. $a^2x^2 - (b + c)^2 = 0.$
35. $z^2 - a^2 - 2ab - b^2 = 0.$
36. $z^4 - 13z^2 + 36 = 0.$
37. $(x + \frac{1}{2})^2 - (2x - \frac{1}{5})^2 = 0.$
38. $(x^2 + 2x)^2 - (x^2 - 4x)^2 = 0.$
39. $4x^4 - 8x^3 - 5x^2 = 0.$
40. $x^3 - 3x^2 + 3x - 1 = 0.$

107. Fractional Equations. Certain fractional equations may be reduced to the linear form or to the form discussed in the last section.

Fractional equations are made integral by clearing of fractions.

The common multiple used in clearing of fractions will contain the variable. It may give an integral equation which is not equivalent to the given fractional equation.

(1) $$x - \frac{1}{x} = 0.$$

Clearing of fractions by multiplying by x, we have

$$x^2 - 1 = 0,$$

or $$(x-1)(x+1) = 0.$$

Whence, $x = 1$ and $x = -1$.

These roots both satisfy the given equation.

When $x = 1$, $x - \frac{1}{x}$ becomes $1 - \frac{1}{1} = 1 - 1 = 0$.

When $x = -1$, $x - \frac{1}{x}$ becomes $-1 - \frac{1}{-1} = -1 + 1 = 0$.

If in clearing of fractions we multiply by x^2, the resulting equation is

$$x^3 - x = 0,$$

or $$x(x-1)(x+1) = 0.$$

Whence, $x = 0$, $x = 1$, and $x = -1$.

We now have three roots, two of which, 1 and -1, satisfy the given equation, while the other one, 0, does not satisfy it; for, when $x = 0$, $x - \frac{1}{x}$ becomes $0 - \frac{1}{0}$, which is not equal to 0.

The root 0 which is here introduced by clearing of fractions is called an *extraneous root.*

The root 0 occurs because we multiplied by a multiple higher than the L. C. M.

In integral equations any multiple whatever of the denominators may be used in clearing of fractions, but in fractional equations the L. C. M. should always be used.

(2) $$5+\frac{x+1}{x^2-1}=4.$$

$$5x^2-5+x+1=4x^2-4.$$

$$5x^2-4x^2+x-5+1+4=0.$$

$$x^2+x=0.$$

$$x(x+1)=0.$$

$$x=0 \text{ and } x=-1.$$

When $x=0$,

$$5+\frac{x+1}{x^2-1}=4 \text{ becomes } 5+\frac{0+1}{0-1}=4, \text{ or } 5-1=4.$$

When $x=-1$,

$$5+\frac{x+1}{x^2-1}=4 \text{ becomes } 5+\frac{-1+1}{1-1}=4, \text{ or } 5+\frac{0}{0}=4.$$

$x=-1$ does not satisfy the equation and is therefore an *extraneous root.*

The root -1 occurs in this solution because the fraction $\frac{x+1}{x^2-1}$ was not reduced to its lowest terms. By reducing it to its lowest term the equation becomes

$$5+\frac{1}{x-1}=4.$$

$$5x-5+1=4x-4.$$

$$5x-4x-5+1+4=0.$$

$$x=0.$$

No *extraneous root* now appears.

Before beginning the solution of a fractional equation, all fractions should be reduced to their lowest terms. The safe

plan in all fractional equations is to test every root, retain only those roots that satisfy the equations, and reject all others as extraneous.

EXERCISES.

1. $\frac{x}{3}-\frac{3}{x}=0.$

2. $4-\frac{3}{x}+\frac{1}{x^2}=0.$

3. $\frac{2}{x}+\frac{1}{x-2}=0.$

4. $\frac{3}{x+1}-\frac{2}{x-1}=0.$

5. $\frac{-2}{x-3}+\frac{1}{x-2}+\frac{1}{x+2}=0.$

6. The quotient of a number divided by 7 increased by the quotient of 63 divided by the number is 6. What is the number?

7. A number is increased by 82 and the sum divided by the number; the quotient is $\frac{1}{14}$ of 1 more than the number. What is the number?

8. The sum of the squares of two consecutive numbers is 85. What are the numbers?

SOLUTION.

Let x = one of the numbers.

$$x+1 = \text{the other.}$$
$$x^2+(x+1)^2=85.$$
$$x^2+x^2+2x+1=85.$$
$$2x^2+2x+1-85=0.$$
$$2x^2+2x-84=0.$$
$$x^2+x-42=0, \quad \text{by dividing by 2.}$$
$$(x+7)(x-6)=0.$$
$$x=-7 \text{ or } x=6.$$

The numbers are 6 and 7 or -7 and -6.

9. The sum of the squares of two consecutive numbers is 41. What are the numbers?

10. Two numbers differ by 5, and their squares differ by 105. What are the numbers?

11. Three times the product of two consecutive numbers lacks 92 of being twice the sum of their squares. What are the numbers?

12. The area of a square field is doubled by increasing its length 12 rods and its width 5 rods. What is the length of one side of the field?

SOLUTION.

Let $x =$ one side of the field.

$$2x^2 = (x + 12)(x + 5).$$
$$2x^2 = x^2 + 17x + 60.$$
$$2x^2 - x^2 - 17x - 60 = 0.$$
$$x^2 - 17x - 60 = 0.$$
$$(x - 20)(x + 3) = 0.$$
$$x = 20 \text{ and } x = -3.$$

Both of these roots satisfy the equation, but only 20 can be used in this problem, as it would not be possible to have a field one side of which is -3 rods in length.

13. The denominator of a fraction is 3 more than its numerator. If 7 is added to each of its terms, the value of the fraction is increased by $\frac{3}{14}$. Find the fraction.

14. A can do a piece of work in 10 days, B in 8 days, and C in 6 days. In how many days can they all do it working together?

SOLUTION.

Let $x =$ the time required.

$$\frac{1}{x} = \text{part done in one day.}$$
$$\frac{1}{10} = \text{part done in one day by A.}$$
$$\frac{1}{8} = \text{part done in one day by B.}$$
$$\frac{1}{6} = \text{part done in one day by C.}$$
$$\frac{1}{10} + \frac{1}{8} + \frac{1}{6} = \text{part done in one day by A, B, and C.}$$
$$\frac{1}{10} + \frac{1}{8} + \frac{1}{6} = \frac{1}{x}.$$
$$12x + 15x + 20x = 120.$$
$$47x = 120.$$
$$x = 2\tfrac{26}{47}, \text{ the number of days required.}$$

15. A cistern has two pipes; one will fill it in 8 hours, and the other in 12 hours. If both are open, how long will the cistern be in filling?

16. A cistern has three pipes; one will fill it in 12 hours, one in 10 hours, and the other will empty it in 15 hours. If all three are open, how long will the cistern be in filling?

17. A number added to 22 times its reciprocal makes 13. Find the number.

$\frac{1}{x}$ is called the Reciprocal of x.

18. A can do a piece of work in a days, B can do it in b days. In how many days can they together do the work?

19. The area of a square field is doubled by increasing its length a rods and its width b rods. What is the length of one side of the field?

20. A fraction whose numerator is 3 less than its denominator added to its reciprocal gives $2\frac{9}{10}$. Find the fraction.

CHAPTER X.

LINEAR EQUATIONS IN TWO VARIABLES.

108. Roots of a Linear Equation in Two Variables. The type form of the linear equation in x and y is

$$ax + by + c = 0.$$

a, b, and c may be any positive or negative numbers whatever.

The equation $2x + 5y - 10 = 0$ is a special form of $ax + by + c = 0$; in which $a = 2$, $b = 5$, and $c = -10$.

If in $2x + 5y - 10 = 0$, we transpose $5y - 10$ and divide by 2, we have

$$x = \frac{10 - 5y}{2}.$$

The value of x depends upon y. It has *one, and only one*, value for each value of y.

If $y = 0$, $x = 5$.
$y = 1$, $x = \frac{5}{2}$.
$y = 2$, $x = 0$.
$y = 3$, $x = -\frac{5}{2}$.
$y = 4$, $x = -5$.
$y = 5$, $x = -\frac{15}{2}$.

If $y = -1$, $x = \frac{15}{2}$.
$y = -2$, $x = 10$.
$y = -3$, $x = \frac{25}{2}$.
$y = -4$, $x = 15$.
$y = -5$, $x = \frac{35}{2}$.
$y = -6$, $x = 20$.

The equation $2x + 5y - 10 = 0$ is satisfied by $x = 5$ and $y = 0$, for these values reduce the equation to

$$2 \times 5 + 5 \times 0 - 10 = 0.$$

$x = \frac{5}{2}$ and $y = 1$ also satisfy the equation, for they reduce it to

$$2 \times \tfrac{5}{2} + 5 \times 1 - 10 = 0.$$

Therefore, $x = 5$, $y = 0$ and $x = \frac{5}{2}$, $y = 1$ are roots of the equation $2x + 5y - 10 = 0$.

In the set of values of x and y above, every value of x and the corresponding value of y constitute a *root*. The number of such sets of values that are possible is evidently unlimited. Besides the roots $(5, 0)$, $(\frac{5}{2}, 1)$, $(0, 2)$, $(-\frac{5}{2}, 3)$, $(-5, 4)$, $(-\frac{15}{2}, 5)$, $(\frac{15}{2}, -1)$, $(10, -2)$, and $(\frac{25}{2}, -3)$, any number more could be worked out at pleasure. A root of an equation in two variables may be written (m, n); m is the value of x, and n is the corresponding value of y, the two together constituting a *root*.

109. Graph of the Linear Equation. The *coördinate axes*, or lines of reference, are two lines perpendicular to each other.

The *axis of abscissas*, or x-line, is the horizontal line $X'OX$.

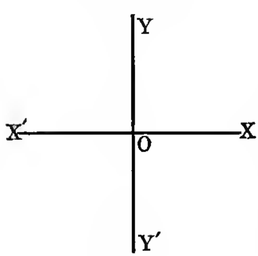


FIG. 1.

The *axis of ordinates*, or y-line, is the vertical line $Y'OY$.

Abscissas, or x-distances, are always measured parallel to the x-line. They are *positive* when measured to the right of the y-line and *negative* when measured to the left of it. *Ordinates*, or y-distances, are always

measured parallel to the y-line. They are *positive* when measured above the x-line and *negative* when measured below it.

The *coördinates* of a point are its x and y distances. The x distance is the *abscissa*, and the y distance the *ordinate*. The coördinates of a point completely determine it with respect to the lines of reference.

A *point* is designated by its coördinates written (m, n). This means that m is the abscissa and n the ordinate of the point.

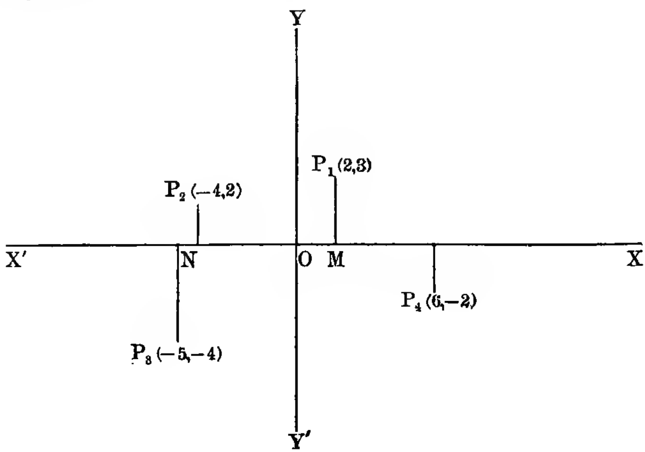


FIG. 2.

The point P_1 or (2, 3) is located by measuring from O to M, a distance of 2, and from M to P_1, parallel to OY, a distance of 3. The point P_2 or $(-4, 2)$ is found by measuring from O to N, a distance of 4, and then from N to P_2, parallel to OY, a distance of 2. The abscissa in this case is measured to the left of OY because it is nega-

tive. The location of the points P_3 or $(-5\ -4)$ and P_4 or $(6, -2)$ is also shown on Figure 2.

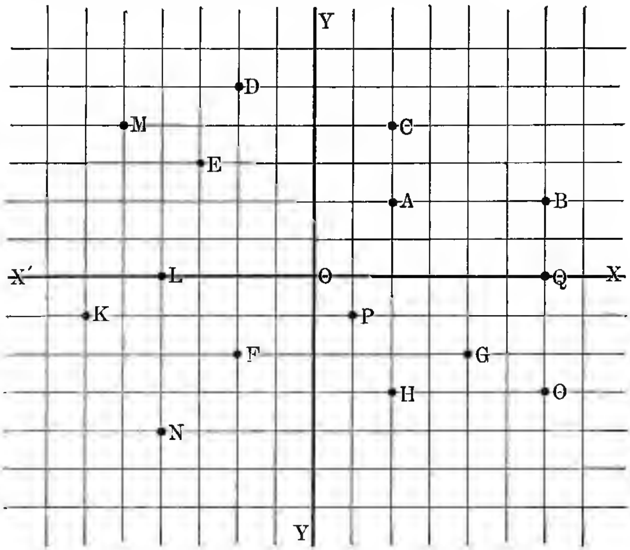


FIG. 3.

EXERCISES.

In Figure 3 the side of each small square is a unit or one.

1. Write the coördinates of each lettered point. Thus, A is $(2, 2)$, K is $(-6, -1)$.

2. Locate on Figure 3 the following points: $(5, 1)$, $(-2, 1)$, $(3, -4)$, $(-3, -2)$, $(6, 1)$, $(-1, -4)$, $(-3, 2)$, $(0, 3)$, $(-3, 0)$, $(0, 0)$, $(1\frac{1}{2}, 3)$, $(2\frac{1}{2}, 2\frac{1}{2})$, $(-3\frac{1}{2}, \frac{1}{2})$.

On Figure 4 are located the points $(10, -2)$, $(7\frac{1}{2}, -1)$, $(5, 0)$, $(2\frac{1}{2}, 1)$, $(0, 2)$, $(-2\frac{1}{2}, 3)$, $(-5, 4)$, $(7\frac{1}{2}, 5)$.

It will be seen that these points are in a straight line. The points located on Figure 4 are some of the roots of the equation $2x + 5y - 10 = 0$, worked out in Section 108.

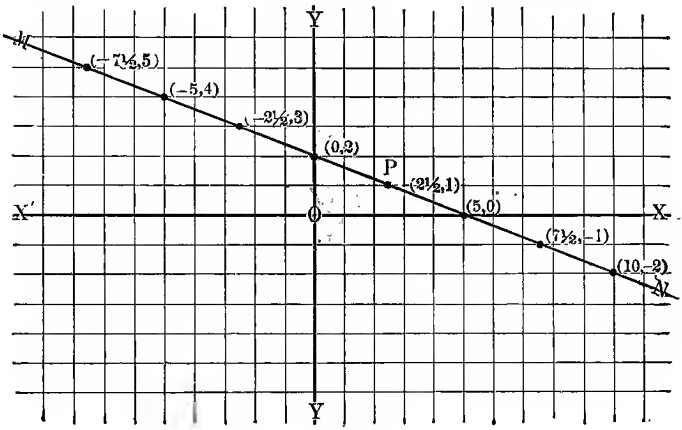


FIG. 4.

The graph of an equation is the line upon which are found all the points indicated by its roots.

The *line MN* is the *graph* of the equation

$$2x + 5y - 10 = 0.$$

The coördinates of every point upon this line satisfy the equation $2x + 5y - 10 = 0$. The point P is $(2, 1\frac{1}{5})$. $(2, 1\frac{1}{5})$ is a root of $2x + 5y - 10 = 0$, for when $x = 2$ and $y = 1\frac{1}{5}$, the equation becomes $2 \times 2 + 5 \times 1\frac{1}{5} - 10 = 0$, which is an identity.

The graph of $x - 2y = 4$. Here $x = 4 + 2y$.

$x = 4$	when	$y = 0.$	$x = 2$	when	$y = -1.$
$x = 6$	when	$y = 1.$	$x = 0$	when	$y = -2.$
$x = 8$	when	$y = 2.$	$x = -2$	when	$y = -3.$

The points represented by the roots above worked out are $(4, 0)$, $(6, 1)$, $(8, 2)$, $(2, -1)$, $(0, -2)$, and $(-2, -3)$ and are shown on Figure 5.

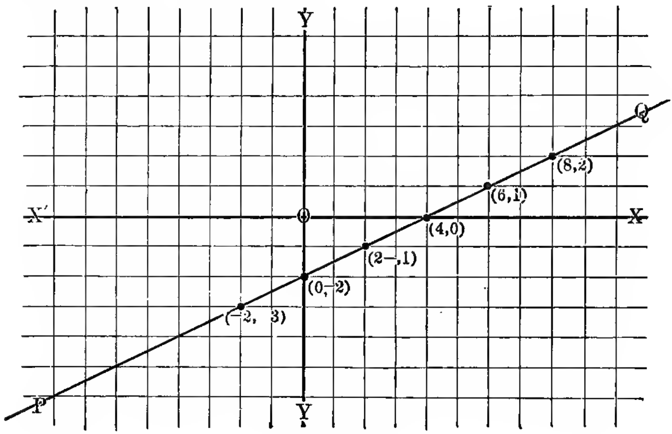


FIG. 5.

The points located in Figure 5 lie upon the straight line PQ, which is the graph of $x - 2y = 4$.

The graph of $x + y + 5 = 0$. Here $x = -(y + 5)$.

$x = -5$ when $y = 0$. $x = 0$ when $y = -5$.

$x = -6$ when $y = 1$. $x = 1$ when $y = -6$.

$x = -2$ when $y = -3$.

The points represented by the above roots are $(-5, 0)$, $(-6, 1)$, $(-2, -3)$, $(0, -5)$, and $(1, -6)$.

Locating these points upon Figure 6, we find that they all lie upon the straight line RS, which is the graph of $x + y + 5 = 0$. This line passes through the points

$A(-4, -1)$, $B(-3, -2)$, and $C(-1, -4)$. Verify that these are roots of the equation.

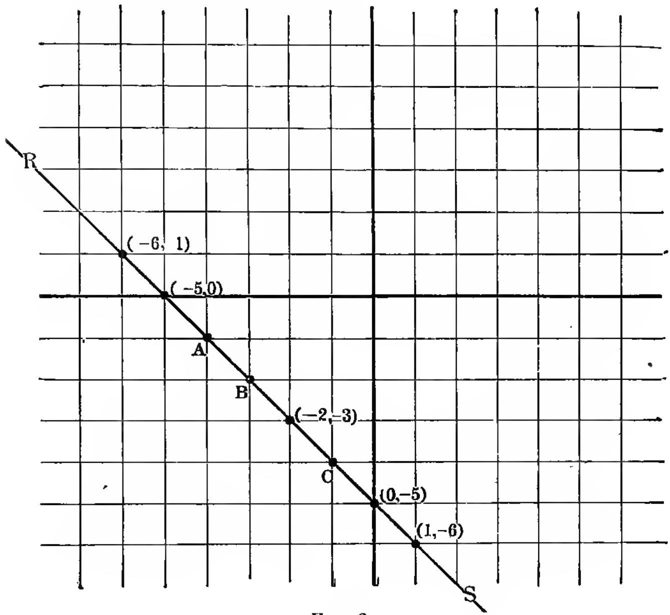


FIG. 6.

EXERCISES.

Draw the graphs of the following equations:

1. $3x - 2y = 6$.
2. $4x - y = 8$.
3. $2x - 5y = 10$.
4. $x - y = 4$.
5. $x + y = 4$.
6. $2x - 3y = 0$.

The graph of every linear equation in two variables is a straight line.

Since two points are sufficient to locate a straight line, we need but two roots of an equation to draw its graph.

The graph of $3x + 5y = 15$.

Since $x = 0$ when $y = 3$, one root is (0, 3);

and since $x = 5$ when $y = 0$, another root is (5, 0).

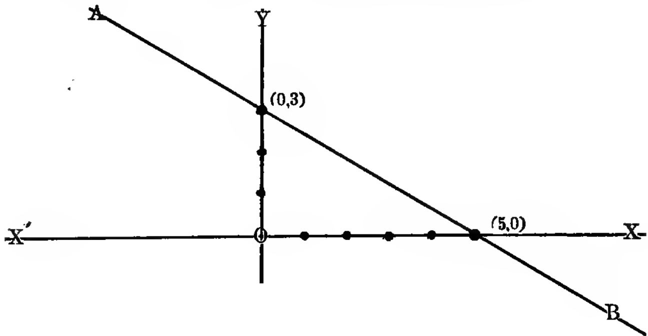


FIG. 7.

Locate these two points upon Figure 7, and draw through them the straight line AB. This line is the graph of $3x + 5y = 15$.

In general, the most convenient pair of roots for determining the graph of an equation is found by making $x = 0$ and solving for y, and then by making $y = 0$ and solving for x. These two roots give the points in which the line cuts the coördinate axes.

EXERCISES.

By the above method make the graphs of the following equations:

1. $3x - 2y = 6$.
2. $4x - y = 8$.
3. $2y - 5x = 10$.
4. $7x - y = 7$.
5. $x + 7y = 7$.
6. $3x + 4y = 12$.

110. Graphs of Two Linear Equations upon the Same Diagram.

Graphs of $x - y = 6$ and $2x + y = 9$.

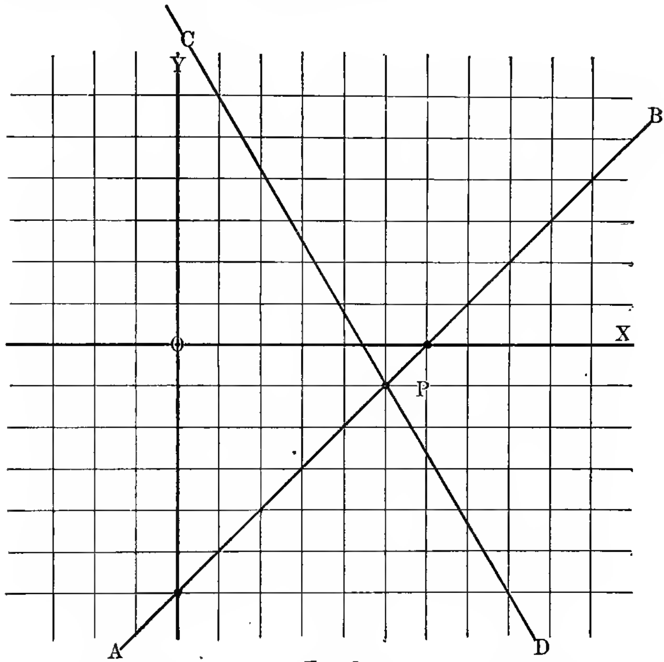


FIG. 8.

The graphs of these lines are AB and CD, respectively. They intersect at the point P $(5, -1)$. This point P lies on both lines, and its coördinates constitute a root of each equation. By putting $x = 5$ and $y = -1$, $x - y = 6$ becomes $5 - (-1) = 6$, and $2x + y = 9$ becomes $2 \times 5 - 1 = 9$. This verifies that $(5, -1)$ is a root of each equation.

Since two straight lines can intersect in but one point, a pair of linear equations can have but one common root.

EXERCISES.

By means of their graphs find the common root of each of the following pairs of equations:

1. $\begin{cases} 2x - y = 1, \\ x + y = 5. \end{cases}$

2. $\begin{cases} x + y = 4, \\ x - 2y = 1. \end{cases}$

3. $\begin{cases} 2x + y = 3, \\ x - 2y = 4. \end{cases}$

4. $\begin{cases} x - 2y = 1, \\ x + y = -5. \end{cases}$

5. $\begin{cases} 2x + 4y = 6, \\ x + y = 1. \end{cases}$

6. $\begin{cases} x - y = 1, \\ 2x + 2y = 9. \end{cases}$

7. Graphs of $x + y = 1$ and $2x + 2y = 9$.

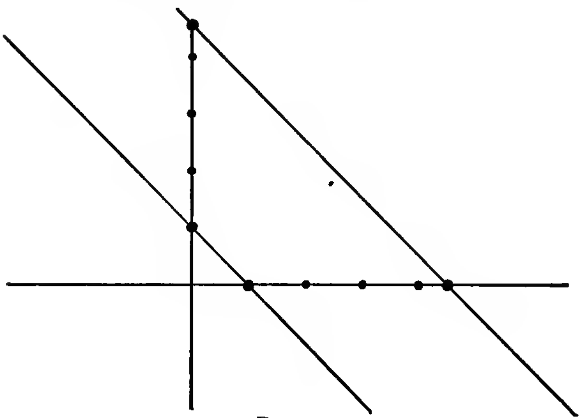


FIG. ..

The graphs of these equations are shown in Figure 9. They are parallel lines, and so do not intersect. The two equations have no common, finite root. It should be noticed that the coefficients of x and y in the second equation are just double the coefficients of x and y in the first equation.

8. $\begin{cases} x - 3y = 6, \\ 3x - 9y = 9. \end{cases}$

9. $\begin{cases} x + 2y = 6, \\ 2x - y = 2. \end{cases}$

CHAPTER XI.

SIMULTANEOUS EQUATIONS.

111. Definitions. *Equations in two or more variables, having the same solutions, are called equivalent when any one of the equations may be changed to the exact form of the others.*

Thus,
$$x + y = 5$$
and
$$3x + 3y = 15$$
are *equivalent equations;* each has a root (1, 4), and the first may be changed to the second by multiplying both members by 3.

Equations not equivalent, but having the same solutions are called simultaneous equations.

Thus,
$$x + y = 6$$
and
$$3x + y = 12$$
are simultaneous, having the common solution $x=3$, $y=3$. These two equations are not equivalent, since the first can not be changed to the form of the second.

Two equations in two variables form a *set* of *simultaneous equations;* three equations in three variables form a set of simultaneous equations, etc.

112. Elimination. To solve a set of simultaneous equations, we must so operate upon and combine the given

equations as to produce a single equation containing a single variable. The processes of obtaining such a single equation are called *elimination.*

In the operations of elimination the following principles are to be carefully noted:

(1) For any expression in an equation an identical expression may be substituted.

(2) When both members of an integral equation are multiplied by an integral expression containing the variable, the resulting equation is not equivalent to the original equation.

For example: $x + 3y = 4.$

Multiplying both sides by $x - y$, we have

$$(x - y)(x + 3y) = 4(x - y),$$

an equation which not only has all the roots of $x + 3y = 4$, but also all the roots of $x - y = 0$.

(3) All the axioms heretofore given can be used in the processes of elimination, with the single exception noted above.

113. Elimination by Substitution. This method will be understood by noting the following solution:

$$\begin{cases} (1) \quad x + 2y = 5, \\ (2) \quad 5x - 2y = 1. \end{cases}$$

By transposing the $2y$ in Equation (1), we have

(3) $$x = 5 - 2y.$$

Substituting this value of x in Equation (2),

(4) $$5(5 - 2y) - 2y = 1.$$

Removing the parentheses,

(5) $25 - 10\,y - 2\,y = 1.$

(6) $-10\,y - 2\,y = 1 - 25,$ by transposing.

(7) $-12\,y = -24,$ by collecting.

(8) $y = 2,$ by dividing.

Substituting this value of y in (3),

(9) $x = 5 - 2(2).$

(10) $x = 1.$

The root is (1, 2).

EXERCISES.

Solve the following equations by the method of substitution:

1. $\begin{cases} x + y = 4, \\ 3\,x - 4\,y = 5. \end{cases}$

2. $\begin{cases} 2\,x + 3\,y = 11, \\ 2\,x + y = 9. \end{cases}$

3. $\begin{cases} 2\,x + 3\,y = 2, \\ x + 3\,y = 1\frac{1}{2}. \end{cases}$

4. $\begin{cases} 3\,x - 2\,y = 5, \\ 6\,x + y = 20. \end{cases}$

5. $\begin{cases} 5\,x + 6\,y = 16, \\ -8\,x + 3\,y = -13. \end{cases}$

6. $\begin{cases} -2\,x + 2\,y = 10, \\ +3\,x + 3\,y = -15. \end{cases}$

7. $\begin{cases} x + 4\,y = 10, \\ 3\,x - y = 12. \end{cases}$

8. $\begin{cases} 2\,x - 7\,y = -5, \\ 3\,x + 2\,y = 7. \end{cases}$

9. $\begin{cases} \dfrac{x}{2} - \dfrac{y}{5} = 1, \\ \dfrac{x}{6} + \dfrac{y}{15} = \dfrac{5}{3}. \end{cases}$

10. $\begin{cases} \frac{2}{3}\,x - \frac{3}{5}\,y = -5, \\ \frac{5}{6}\,x + \frac{4}{5}\,y = 17. \end{cases}$

Construct graphs for Exercises 4, 5, 6, and 7.

114. Elimination by Comparison. The following problem will illustrate the method:

$$\begin{cases} (1) \qquad 5x+3y=19, \\ (2) \qquad 4x-y=5. \end{cases}$$

By transposing $3y$ and dividing by 5, we get from (1)

$$(3) \qquad x=\frac{19-3y}{5}.$$

By transposing $-y$ and dividing by 4, we get from (2)

$$(4) \qquad x=\frac{5+y}{4}.$$

Equating the two values of x given by (3) and (4),

$$(5) \qquad \frac{19-3y}{5}=\frac{5+y}{4}.$$

(6) $76-12y=25+5y$, by clearing of fractions.

(7) $-12y-5y=25-76$, by transposing.

(8) $-17y=-51$, by collecting.

(9) $y=3$.

Substituting this value of y in (3),

$$(10) \qquad x=\frac{19-3\times 3}{5}=\frac{10}{5}=2.$$

The root is (2, 3).

EXERCISES.

Solve by the method of comparison:

1. $\begin{cases} 2x-3y=7, \\ 2x+y=3. \end{cases}$

2. $\begin{cases} 5x+y=3, \\ 2x-3y=-4. \end{cases}$

3. $\begin{cases} 4x+y=3, \\ 2x+3y=4. \end{cases}$

4. $\begin{cases} 3x-4y=4, \\ 2x+5y=10. \end{cases}$

5. $\begin{cases} 5x - 2y = 4, \\ 3x + 7y = 21. \end{cases}$

6. $\begin{cases} x - y = a, \\ x + 2y = b. \end{cases}$

7. $\begin{cases} 5x - 4y = 20, \\ 3x + 2y = 12. \end{cases}$

8. $\begin{cases} -4x - 2y = 1, \\ -x + 5y = 6. \end{cases}$

9. $\begin{cases} x + 2y = 4, \\ 5x + y = 3. \end{cases}$

10. $\begin{cases} 5x + y = m, \\ 3x + 4y = 4m. \end{cases}$

Construct graphs for Exercises 3, 5, 7, and 9.

115. Elimination by Addition or Subtraction. The two problems here solved will illustrate the method.

1. Solve $\begin{cases} (1) & 2x + 3y = 5, \\ (2) & 7x - 2y = 5. \end{cases}$

We first make the coefficients of the y's have the same absolute value. This is done by multiplying Equation (1) by 2 and Equation (2) by 3, thus giving us

(3) $$4x + 6y = 10,$$

(4) $$21x - 6y = 15.$$

Adding Equations (3) and (4),

(5) $$25x = 25.$$

(6) $$x = 1.$$

Substituting this value of x in (1),

(7) $$2 \times 1 + 3y = 5.$$

(8) $$3y = 5 - 2 = 3.$$

(9) $$y = 1.$$

The root is (1, 1).

2. Solve $\begin{cases} (1)\ 2x + 7y = 38. \\ (2)\ 3x + 4y = 31. \end{cases}$

We can make the coefficients of the x's alike by multiplying Equation (1) by 3 and Equation (2) by 2, thus giving

(3) $$6x + 21y = 114,$$

(4) $$6x + 8y = 62.$$

Subtracting Equation (4) from (3),

(5) $$13y = 52.$$

(6) $$y = 4.$$

Substituting this value of y in Equation (1),

(7) $$2x + 7 \cdot 4 = 38.$$

(8) $$2x = 38 - 28 = 10.$$

(9) $$x = 5.$$

The root is (5, 4).

The method of elimination generally used is that of addition or subtraction. The method by comparison is merely a disguised form of eliminating by subtraction. The particular method to be used must be determined by a careful inspection of the problem.

116. Some Illustrative Examples.

1. $\begin{cases} (1)\ \dfrac{x}{4} + \dfrac{y}{3} = 4, \\ (2)\ \dfrac{x}{2} - \dfrac{y}{3} = 2. \end{cases}$

Here there is no need of clearing of fractions. By addition, y is eliminated, and

(3) $$\frac{3x}{4} = 6.$$

(4) $$3x = 24.$$

(5) $$x = 8.$$

Substituting 8 for x in (1), we have

(6) $$\frac{8}{4} + \frac{y}{3} = 4.$$

(7) $$\frac{y}{3} = 4 - 2 = 2.$$

(8) $$y = 6.$$

The root is (8, 4).

2. $$\begin{cases} (1) & \dfrac{3}{x} + \dfrac{2}{y} = \dfrac{17}{24}, \\ (2) & \dfrac{5}{x} - \dfrac{1}{y} = \dfrac{11}{24}. \end{cases}$$

In problems of this form never clear of fractions.

(3) $$\frac{10}{x} - \frac{2}{y} = \frac{22}{24},$$ by multiplying (2) by 2.

(4) $$\frac{13}{x} = \frac{39}{24},$$ by adding (1) and (3).

(5) $$39x = 13 \times 24.$$

(6) $$x = 8.$$

Substituting 8 for x in (1),

(7) $$\frac{3}{8} + \frac{2}{y} = \frac{17}{24}.$$

(8) $$\frac{2}{y} = \frac{17}{24} - \frac{3}{8} = \frac{1}{3}.$$

(9) $$y = 6.$$

The root is (8, 6).

3. $\begin{cases} (1) \quad x - 3y = 10, \\ (2) \quad 3x + 5y = 2. \end{cases}$

In this form, in which one equation has the variable with a coefficient 1, use the method of substitution.

From (1),

(3) $\qquad x = 3y + 10.$

Substituting the value of x in (2),

(4) $3(3y+10)+5y=2.$

(5) $\quad 9y+30+5y=2.$

(6) $\qquad 14y = -28$, transposing and combining.

(7) $\qquad y = -2.$

(8) $\qquad x = -6+10=4$, from (3).

The root is $(4, -2)$.

4. $\begin{cases} (1) \quad \dfrac{1+2x}{3y} + 1 = \dfrac{x}{y}, \\ (2) \quad \dfrac{y}{x} + 1 = \dfrac{5}{x}. \end{cases}$

In examples of this form it is best to clear of fractions.

(3) $1 + 2x + 3y = 3x$, by clearing (1) of fractions.

(4) $\quad -x + 3y = -1$, by transposing and collecting.

(5) $\quad x + y = 5$, by clearing (2) of fractions.

(6) $\quad 4y = 4$, by adding (4) and (5).

(7) $\quad y = 1.$

(8) $\quad x = 4$, by substituting 1 for y in (4).

The root is $(4, 1)$.

EXERCISES.

Solve the following simultaneous equations:

1. $\begin{cases} x + y = 5, \\ 2x + 3y = 12. \end{cases}$

2. $\begin{cases} 2x - 5y = 12, \\ 3x + 5y = 8. \end{cases}$

3. $\begin{cases} 4x + 8y = 13, \\ -2x + 3y = 7\frac{1}{2}. \end{cases}$

4. $\begin{cases} \frac{3}{4}x - \frac{5}{6}y = 1, \\ \frac{2}{3}x + \frac{1}{2}y = 4. \end{cases}$

5. $\begin{cases} 4x - \dfrac{y}{5} = 6, \\ \dfrac{x}{4} + \dfrac{y}{3} = 3\frac{5}{6}. \end{cases}$

6. $\begin{cases} \dfrac{5}{x} - \dfrac{4}{y} = -1, \\ \dfrac{8}{x} + \dfrac{2}{y} = \dfrac{13}{5}. \end{cases}$

7. $\begin{cases} ax + by = c, \\ lx + my = p. \end{cases}$

8. $\begin{cases} x + \dfrac{1}{y} = 5, \\ 3x - \dfrac{5}{y} = 7. \end{cases}$

9. $\begin{cases} \dfrac{3}{x} + \dfrac{7}{y} = 7, \\ \dfrac{2}{x} - \dfrac{5}{y} = 5. \end{cases}$

10. $\begin{cases} \dfrac{2}{x} - \dfrac{3}{y} = 11, \\ \dfrac{3}{x} + \dfrac{1}{y} = 22. \end{cases}$

11. $\begin{cases} x + \dfrac{2}{y} = 5, \\ 3x - \dfrac{2}{y} = 3. \end{cases}$

12. $\begin{cases} lx + \dfrac{m}{y} = a, \\ kx - \dfrac{n}{y} = b. \end{cases}$

13. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = c, \\ \dfrac{b}{x} + \dfrac{a}{y} = b. \end{cases}$

14. $\begin{cases} x + y = m + n, \\ \dfrac{m + x}{n + y} = \dfrac{n}{m}. \end{cases}$

15. $\begin{cases} 3x - 7y = 0, \\ \frac{2}{7}x + \frac{5}{3}y = 7. \end{cases}$

16. $\begin{cases} ax - by = 0, \\ mx + ny = q. \end{cases}$

17. $\begin{cases} mx + ny = m^2, \\ nx + my = n^2. \end{cases}$

18. $\frac{x}{3}+\frac{y}{4}=3x-7y-37=0.$ **19.** $\frac{x}{a}+\frac{y}{b}=1=\frac{x-a}{b}+\frac{y-b}{a}.$

20. $$\begin{cases} \dfrac{3y+7x+1}{5}-\dfrac{2y-3x+8}{3}=2, \\ \dfrac{5y-7x+10}{3}-\dfrac{3y+2x+6}{5}=2. \end{cases}$$

Construct graphs for Exercises 2 and 5.

EXERCISES.

1. The sum of two numbers is 32, and one number is 3 times the other. What are the numbers?

Solution.

Let x and y be the numbers.

Then (1) $x+y=32$,

and (2) $x=3y$, by the conditions of the problem.

(3) $3y+y=32.$

(4) $4y=32.$

(5) $y=8.$

(6) $x=24.$

2. Eight apples and 5 oranges cost 31 cents, and 5 apples and 10 oranges cost 40 cents. What is the cost of 1 apple and of 1 orange?

3. Three bushels of wheat cost 15 cents more than 5 bushels of corn, and 2 bushels of wheat and 1 bushel of corn together cost \$2.05. What is the price per bushel of each?

4. A fraction is equal to $\frac{3}{4}$. If both of its terms are increased by 12, the value is then $\frac{4}{5}$. Find the fraction.

(Let x = numerator, y = denominator, $\frac{x}{y}$ = fraction.)

5. Find a fraction such that if 1 is added to the numerator it becomes $\frac{1}{2}$, and if 5 is added to the denominator it becomes $\frac{1}{3}$.

6. The sum of two numbers is 75. The larger contains the smaller 5 times, with a remainder of 3. Find the number.

7. There are two numbers; 3 times the first is 8 more than the second, and their difference is 42. Find them.

8. A man spent $225 for sheep at $3.50 a head and calves at $10 a head. He bought 42 head in all. How many of each did he buy?

9. Ten years ago a father was 5 times as old as his son. Twenty years hence he will be twice as old. What are the present ages of each?

10. A said to B, "Give me $60, and I shall have twice as much as you." B said to A, "Give me $90, and I shall have as much as you." How much had each?

11. Find two numbers such that $\frac{1}{2}$ the first and $\frac{1}{3}$ the second is 36, and $\frac{1}{9}$ the first and $\frac{1}{6}$ the second is 13.

12. There are two numbers such that if each is increased by 5, the sums are in the ratio 5 and 11, and if each number be decreased by 15, the remainders are in the ratio 1 and 7. Find the numbers.

13. A farmer has two horses and an $18 saddle. If the saddle is put on the cheaper horse, the horse and saddle are worth $\frac{3}{5}$ of the better horse. The better horse and saddle lack $12 of being worth twice as much as the cheaper horse. What is the value of each horse?

14. If the greater of two sums be multiplied by 5 and the lesser by 7, the sum of the products is 140. If the greater be divided by 7 and the lesser by 5, the difference of the quotients is 0. Find the numbers.

15. There are two numbers which differ by 11. One sixth of the larger is 1 more than $\frac{1}{5}$ of the smaller. Find the numbers.

16. A number consists of two digits whose sum is 13; if 27 be added to the number, the order of the digits is changed. Find the number.

SOLUTION.

Let $x =$ units' digit.

$y =$ tens' digit.

$10y + x =$ the number.

Then (1) $x + y = 13,$

and (2) $10y + x + 27 = 10x + y,$ by conditions of problem.

(3) $9y - 9x = -27.$

(4) $y - x = -3.$

(5) $2y = 10,$ by adding (1) and (4).

(6) $y = 5.$

(7) $x = 8,$ from (1).

$10y + x = 58,$ the number.

17. If to a certain number of two digits the tens' digit be added, the sum is 80. If the units' digit be subtracted, the remainder is 70. Find the number.

18. A number is composed of two digits whose sum is 13. If their order is inverted, the new number is 4 less than double the original number. Find the number.

19. A sum of money was divided equally among a certain number of people. If there had been 3 persons more, the share of each would have been \$2 less; but if there had been 2 persons fewer, the share of each would have been \$2 more. How many persons were there, and what was the share of each?

20. A lost $\frac{3}{4}$ of his money and then borrowed $\frac{1}{6}$ of B's money, when he had \$12. At first A had $\frac{2}{3}$ as much as B. Find how much each had at first.

21. The sum of a number of two digits and the number formed by reversing the order of the digits is 110. The difference of the digits is 8. Find the number.

22. A man has a certain number of silver dollars and quarters. He notices that if his dollars were quarters and his quarters dollars he would have \$22.50 more than he now has. He also notices that if his dollars were dimes and his quarters half dollars, he would have \$1 more than he now has. How much money has he?

23. In a certain school $\frac{1}{2}$ of the number of boys is equal to $\frac{1}{3}$ of the number of girls; twice the whole number of pupils in the school is 100 more than 3 times the number of girls. How many pupils in the school?

24. A and B are 45 miles apart. If they walk in the same direction, A overtakes B in 45 hours. If they walk toward each other, they meet in 5 hours. Find their rates of walking.

25. If the first of two numbers be divided by 12 and the second by 15, the sum of the quotients is 12; if the first be divided by 4 and the second by 3, the difference of the quotients is 12. What are the numbers?

26. Find two numbers such that the sum of their reciprocals is $\frac{24}{143}$, and the difference of their reciprocals is $\frac{2}{143}$.

27. If the base of a rectangle be increased by 6 feet and the altitude by 4 feet, the area is increased by 216 square feet. If the base be decreased by 4 feet, and the altitude increased by 4 feet, the rectangle becomes a square. Find the base and altitude of the rectangle.

28. If B loans A \$500, A will then have 3 times as much money as B has left; but if A loans B \$200, B will have twice as much money as A has left. How much money has each?

29. A sum of money on interest amounted to \$824 in 9 months and to \$840 in 15 months. Find the principal and the rate.

30. If the greater of two numbers be divided by the less, the quotient is 1, with a remainder of 8; if 4 times the less be divided by the greater, the quotient is 2 with a remainder of 22. What are the numbers?

31. Sixty workmen, consisting of men and boys, did a piece of work in 5 days and received for it \$ 430. The men were paid \$ 1.75 a day, and the boys 80 cents a day. How many men and how many boys were there?

32. Find a fraction such that when the numerator is trebled and the denominator decreased by 4 the value becomes 3, and when the denominator is trebled and the numerator increased by 4 the value becomes $\frac{1}{3}$.

33. In 10 years I will be 5 times as old as my son was 5 years ago, and 2 years ago I was twice as old as my son will be 4 years hence. Find my age and that of my son.

34. The lengths of two ropes are as 4 : 5, and when 20 feet is cut from each rope the remainders are as 3 : 4. Find the lengths of the ropes.

35. A river flows 3 miles an hour; a boat going down the river passes a certain point in 12 seconds and in going up it takes 18 seconds. Find the speed of the boat in still water and the length of the boat.

117. System of Linear Equations with Three or More Variables. We have seen that in order to solve linear equations with two variables, we must have a set of two independent equations; in like manner, when we have three variables, we must have a set of three independent equations; when four variables, we must have a set of four equations, etc.

The method of solving a problem in three variables will be understood by noting the following solutions:

$$\mathbf{1.}\quad \begin{cases} (1)\ x - 2y + z = 1, \\ (2)\ 3x + y - z = 4, \\ (3)\ 2x + y + z = 12. \end{cases}$$

By looking at this problem we see that the z's can be easily eliminated.

(4) $4x - y = 5,$ by adding (1) and (2).
(5) $5x + 2y = 16,$ by adding (2) and (3).
(6) $8x - 2y = 10,$ by multiplying (4) by 2.
(7) $13x = 26,$ by adding (5) and (6).
(8) $x = 2.$
(9) $y = 3,$ by substituting 2 for x in (4).
(10) $z = 5,$
by substituting 2 for x, 3 for y in (1).

The root is (2, 3, 5).

2. $\begin{cases} (1)\ 4x - 3y + 2z = 3, \\ (2)\ 6x + 3y + 3z = 7, \\ (3)\ 2x - 6y + 5z = 4. \end{cases}$

By adding (1) and (2), we eliminate y, and have

$$10x + 5z = 10,$$

or (4) $$2x + z = 2.$$

Multiplying (1) by 2 and subtracting (3), we have

(5) $$6x - z = 2.$$

Adding (4) and (5),

(6) $$8x = 4.$$

(7) $$x = \tfrac{1}{2}.$$

Substituting $x = \frac{1}{2}$ in (4),

(8) $$1 + z = 2.$$

(9) $$z = 1.$$

Substituting $x = \frac{1}{2}$ and $z = 1$ in (1),

(10) $$2 - 3y + 2 = 3.$$

(11) $$-3y = -1.$$

(12) $$y = \tfrac{1}{3}.$$

The root is $(\frac{1}{2}, \frac{1}{3}, 1)$.

EXERCISES.

1. $\begin{cases} x+z=11, \\ y+z=6, \\ 2x+y=25. \end{cases}$

2. $\begin{cases} 2x+3z=54, \\ 7y+5z=106, \\ 3x+5y=76. \end{cases}$

3. $\begin{cases} 2x-5y+4z=7, \\ 3x-2y+z=5, \\ 5x+3y-5z=2. \end{cases}$

4. $\begin{cases} \dfrac{x}{7}+\dfrac{y}{5}+\dfrac{z}{4}=2, \\ \dfrac{x}{2}+\dfrac{y}{10}-\dfrac{z}{8}=9, \\ x+y+3z=0. \end{cases}$

5. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=9, \\ \dfrac{1}{x}-\dfrac{1}{y}+\dfrac{1}{z}=3, \\ \dfrac{1}{x}+\dfrac{1}{y}-\dfrac{1}{z}=1. \end{cases}$

6. $\begin{cases} \dfrac{2}{x}-\dfrac{3}{y}+\dfrac{1}{z}=-1, \\ -\dfrac{3}{x}+\dfrac{2}{y}+\dfrac{1}{z}=4, \\ \dfrac{5}{x}+\dfrac{1}{y}-\dfrac{3}{z}=1. \end{cases}$

7. $\begin{cases} \dfrac{x}{m}+\dfrac{y}{n}=r, \\ \dfrac{x}{m}+\dfrac{z}{l}=r, \\ \dfrac{y}{n}+\dfrac{z}{l}=r. \end{cases}$

8. $\begin{cases} \dfrac{p}{x}+\dfrac{q}{y}=s, \\ \dfrac{q}{y}+\dfrac{r}{z}=s, \\ \dfrac{r}{z}+\dfrac{p}{x}=s. \end{cases}$

9. $\begin{cases} \dfrac{x}{3}-\dfrac{y}{2}+z=3, \\ \dfrac{x}{2}-\dfrac{y}{4}+z=5, \\ \dfrac{x}{6}+\dfrac{y}{4}-\dfrac{z}{3}=1. \end{cases}$

(Do not clear of fractions in Exercise 5.)

10. $\begin{cases} x+y+z+w=10, \\ 2x-y+3z+w=13, \\ x+3y-2z-w=-3, \\ x-2y+3z-2w=-2. \end{cases}$

11. Three men have together \$750; $\frac{1}{3}$ of A's and $\frac{1}{3}$ of C's is equal to $\frac{1}{2}$ of B's; twice A's is \$150 more than both B's and C's. How much money has each?

12. The sum of three numbers taken two and two are 68, 94, and 62, respectively. Find the numbers.

13. There are three numbers such that the first with $\frac{1}{3}$ the second is 68; $\frac{1}{2}$ the first with $\frac{2}{3}$ the third is 73; and the second with $\frac{1}{2}$ the third is 90. Find the numbers.

14. A number consists of three digits whose sum is 17. The hundreds' digit is 3 times the tens' digit. If the order of the digits be reversed, the number is increased by 297. Find the number.

15. A and B can do a piece of work in 6 days, B and C in $7\frac{1}{2}$ days, and C and A in 10 days. In how many days can each do the work separately?

16. A cistern has three pipes A, B, and C. If A and B fill while C empties, the cistern will be filled in 60 minutes. If A and C fill while B empties, the cistern will be filled in 24 minutes. If B and C fill while A empties, the cistern will be filled in 120 minutes. In what time could each pipe fill it alone?

17. There are three numbers whose sum is 113; $\frac{1}{2}$ the second is 2 more than $\frac{1}{3}$ the third; $\frac{2}{3}$ of the first lacks 3 of being $\frac{1}{2}$ the second. Find the numbers.

18. Separate the number 180 into three parts, such that the second divided by the first equals 2, the third divided by the second equals 3, and the first divided by the third equals $\frac{1}{6}$.

19. A, B, and C have certain sums of money. If A gives B \$100, they will have the same amount; if C gives A \$200, he will have as much left as A and B together then have; if B's money were doubled and A's increased by \$100, they would then have together as much as C. What sum has each?

CHAPTER XII.

EVOLUTION.

118. Definitions. *Square Root.* One of the *two* equal factors of a number is called its *square root.*

$25 = 5 \times 5$, hence 5 is a square root of 25.

Cube Root. One of the *three* equal factors of a number is called its *cube root.*

$64 = 4 \times 4 \times 4$, hence 4 is a cube root of 64.

nth Root. One of the n equal factors of a number is called its *nth Root.*

$a^n = a \times a \times a \times a \cdots$ to n factors, hence a is an nth root of a^n.

Evolution. The process of finding roots is called *evolution.* It is the inverse of *involution.*

From the definition we see that the square root of a^4 is $a^{\frac{4}{2}} = a^2$, the cube root of a^{12} is $a^{\frac{12}{3}} = a^4$, and the nth root of a^m is $a^{\frac{m}{n}}$.

The Radical Sign. When the sign $\sqrt{\ }$ is placed before a number, a root is to be extracted. The number written over the radical sign is called the *index*, and denotes what root is to be extracted. Thus, $\sqrt[2]{16} = 4$, $\sqrt[3]{125} = 5$, $\sqrt[4]{16} = 2$, $\sqrt[5]{32} = 2$, $\sqrt[n]{a^n} = a$; the indices 2, 3, 4, 5, n, denote, respectively, the 2d, 3d, 4th, 5th, and nth roots of the number before which the radical sign ($\sqrt{\ }$) is placed. The square root sign ($\sqrt[2]{\ }$) is usually written $\sqrt{\ }$.

The Radicand. The number whose root is to be extracted is called the *radicand.* Thus, in $\sqrt{25}$, $\sqrt[3]{125}$, 25 and 125 are radicands.

The root of an expression of two or more terms is denoted by the radical sign in connection with the vinculum or parenthesis.

$\sqrt{a^2 + 2\,ab + b^2}$ and $\sqrt{}(a^2 + 2\,ab + b^2)$ each denote the square root of $a^2 + 2\,ab + b^2$.

119. Even and Odd Roots. An *even root* has an even index, an *odd root* an odd index.

$\sqrt{4}$, $\sqrt[4]{8}$, $\sqrt[10]{5}$, are even roots. $\sqrt[3]{8}$, $\sqrt[5]{32}$, $\sqrt[7]{a^{14}}$, are odd roots.

An even number of equal positive or negative factors multiplied together gives a positive product. Hence, only positive numbers can have real even roots. An even root of a negative number is called an *imaginary number.* All other numbers are *real.* All numbers are either *real* or *imaginary*.

5, $\sqrt{7}$, $\sqrt[10]{21}$, a, b, are real numbers. $\sqrt{-1}$, $\sqrt{-5}$, $\sqrt[4]{-16}$, $\sqrt[6]{-a^2}$, are imaginary numbers.

120. The Law of Signs. (1) An even root of a positive number is either + or −.

$$\sqrt{81} = \pm 9, \text{ for } (+9) \times (+9) = 81$$

and

$$(-9) \times (-9) = 81.$$

$$\sqrt{a^2} = \pm a.$$

(2) A negative number has no real even roots.

(3) An odd root of a negative number is −.

$\sqrt[3]{-27} = -3$, for $(-3)(-3)(-3) = -27$. $\sqrt[5]{-32} = -2$.

EXERCISES.

Extract the indicated roots:

1. $\sqrt{25\,a^2} = \pm 5\,a.$
2. $\sqrt{a^4b^2c} = \pm a^2b\sqrt{c}.$
3. $\sqrt{16\,a^6b^2c^4}.$
4. $\sqrt{169\,x^4y^2z^2}.$
5. $\sqrt[3]{1728\,x^3y^6}.$
6. $\sqrt[3]{-8\,a^3b^3c^9} = -2\,abc^3.$
7. $\sqrt{a^2-4\,ab+4\,b^2} = \pm(a-2\,b).$
8. $\sqrt{a^2+6\,ax+9\,x^2}.$
9. $\sqrt[3]{-125\,x^{12}y^3}.$
10. $\sqrt{\dfrac{a^2x^2}{c^2d^2}}$
11. $\sqrt{625\,a^{10}b^{30}c^2}.$
12. $\sqrt[3]{1000\,a^3b^6c^9}.$
13. $\sqrt{400\,a^{30}b^{16}c^{18}}.$
14. $\sqrt{\dfrac{144\,a^2b^2}{(x+y)^2}}.$
15. $\sqrt[3]{-343\,a^9b^{12}c^3}.$
16. $\sqrt{\dfrac{81\,x^4y^8}{256\,a^8b^4c^6}}.$
17. $\sqrt[4]{16\,a^{16}b^{20}c^8}.$
18. $\sqrt[5]{-32\,a^{25}x^{30}y^5}.$
19. $\sqrt[6]{a^{12}b^{30}c^{42}}.$
20. $\sqrt[7]{-a^{21}b^{35}c^{56}}.$

SQUARE ROOT.

121. The Square Root of a Polynomial. The square root of a polynomial is found by the reversal of the method used in squaring a polynomial.

$(A+B)^2 = A^2 + 2\,AB + B^2$ is the general type form of the square of the sum of any two quantities, and is the type form used in the reversal process.

By comparing any perfect square, whose root consists of two terms, with this type, its root may be easily determined.

$$x^2 + 10\,x + 25.$$

This may be written $x^2 + 2\,x \cdot 5 + 5^2$.

It is at once seen that $A = x$ and $B = 5$. Hence, the square root of $x^2 + 10\,x + 25$ is $x + 5$.

The same method may often be applied to polynomials whose roots have three terms.

$$x^2 + y^2 + 9 + 2\,xy + 6\,x + 6\,y.$$

Arrange this in type form, according to x.

$$x^2 + 2\,x(y + 3) + y^2 + 6\,y + 9.$$

This may be further arranged

$$x^2 + 2\,x(y + 3) + (y + 3)^2,$$

and the square root is at once seen to be $x + y + 3$.

EXERCISES.

Find the square root of the following by comparing with the type form:

1. $x^2 + 16\,x + 64$.
2. $9\,x^2 + 24\,x + 16$. (Write $(3\,x)^2 + 2(3\,x)4 + 16$.)
3. $x^2 - 18\,x + 81$.
4. $x^2 - 10\,xy + 25\,y^2$.
5. $16\,x^2 + 56\,xy + 49\,y^2$.
6. $a^2x^2 + 2\,abxy + b^2y^2$.
7. $x^2 + 2\,xy + y^2 + z^2 + 2\,zx + 2\,yz$.
8. $x^2 + 9\,y^2 + z^2 + 6\,xy + 2\,zx + 6\,yz$.
9. $a^2 - 6\,bc + b^2 + 9\,c^2 - 2\,ab + 6\,ca$.
10. $4\,x^2 + y^2 + 9\,z^2 - 4\,xy - 6\,yz + 12\,zx$.
11. $a^2 + b^2 + c^2 + d^2 + 2\,ab + 2\,bc + 2\,ca + 2\,ad + 2\,bd + 2\,cd$.
12. $x^2 - 6\,xy + 9\,y^2 + 16 + 8\,x - 24\,y$.

122. Formal Extraction of Square Root. When the root can not be easily determined by inspection, we reverse the type form.

$$\begin{array}{r|l}
 & A^2 + 2\,AB + B^2 \;\underline{|\,A + B\,} \\
 & A^2 \\
\hline
2\,A + B & 2\,AB + B^2 \\
 & \underline{2\,AB + B^2}
\end{array}$$

The first term of the root A is the square root of A^2. The second term, B, is contained in $2AB$, and is found from it by dividing by $2A$. $2A$ is called the trial divisor. $(2A+B)$ is the complete divisor. When this is multiplied by B, the result is $2AB+B^2$, which is the part of the square remaining after A^2 is subtracted.

(1) Find the square root of $36x^2 - 144xy + 144y^2$.

$$
\begin{array}{r|l|l}
 & 36x^2 - 144xy + 144y^2 & \underline{6x - 12y} \\
 & \underline{36x^2} & \\
12x - 12y & -144xy + 144y^2 & \\
 & \underline{-144xy + 144y^2} &
\end{array}
$$

Here the square root of $36x^2$ is $6x$. $12x$ is the trial divisor, which divided into $-144xy$ gives $-12y$ as the next term of the root. The complete divisor is $12x-12y$, which multiplied by $-12y$ gives $-144xy+144y^2$, the remaining part of the square after $36x^2$ is subtracted. $6x-12y$ is the required square root. This method is easily extended, as the following example will show:

(2) Find the square root of $x^4+6x^3+x^2-24x+16$.

$$
\begin{array}{r|l|l}
 & x^4+6x^3+x^2-24x+16 & \underline{x^2+3x-4} \\
 & \underline{x^4} & \\
2x^2+3x & 6x^3+x^2 & \\
 & \underline{6x^3+9x^2} & \\
2x^2+6x-4 & \qquad -8x^2-24x+16 & \\
 & \qquad \underline{-8x^2-24x+16} &
\end{array}
$$

Note that the polynomial is arranged according to powers of x.

The square root of x^4 is x^2. The trial divisor is $2x^2$, which divided into $6x^3$ gives $3x$, the next term of the

root. The complete divisor is $2x^2+3x$. When this is multiplied by $3x$ it gives $6x^3+9x^2$, which leaves, when subtracted from the remaining part of the square, $-8x^2-24x+16$. (x^2+3x) is now regarded as the first part of the root, giving a trial divisor of $2x^2+6x$. This gives -4 as the next term of the root. The complete divisor is $2x^2+6x-4$. When this is multiplied by -4 it gives $-8x^2-24x+16$, which is the part of the square remaining. The root is x^2+3x-4.

123. Rule for Extracting the Square Root:

(1) *Arrange the terms with respect to the powers of some letter.*

(2) *Extract the square root of the first term, place its root as the first term of the root sought, and take its square from the given polynomial.*

(3) *Double the root already found for a trial divisor, divide the first term of the remainder by the trial divisor, placing the quotient as the next term of the root, and also annexing it to the trial divisor to form the complete divisor.*

(4) *Multiply the complete divisor by the last term of the root, and take the product from the first remainder.*

(5) *Continue this process until the other terms of the root are found.*

EXERCISES.

Extract the square root of the following:

1. $x^4+14x^2+4x^3+20x+25$.
2. $x^4-4x^3+10x^2-12x+9$.
3. $4x^4+25x^2+12x^3+24x+16$.
4. $x^6+1+12x^5+42x^4+6x+38x^3+21x^2$.
5. $4x^4+5x^2y^2+y^4+4x^3y+2xy^3$.

6. $9x^4 + 37x^2y^2 + 4y^4 - 30x^3y - 20xy^3$.

7. $1 - 2x - 6y + x^2 + 6xy + 9y^2$.

8. $4x^2 - 12xy + 16xz + 9y^2 + 16z^2 - 24yz$.

9. $4 + x^6 + 4x^3 - 12x - 6x^4 + 9x^2$.

10. $\dfrac{y^4}{9} + \dfrac{2by^2}{3} + \dfrac{4b^2y^2}{3} + b^3y + \dfrac{b^4}{4}$.

124. Inexact Square Roots. The following example will sufficiently show the method.

Extract the square root of $1 + 3x$ to four terms.

$$\begin{array}{r|l}
 & 1 + 3x \;\big|\; \underline{1 + \tfrac{3}{2}x - \tfrac{9}{8}x^2 + \tfrac{27}{16}x^3} \\
 & \underline{1} \\
2 + \tfrac{3}{2}x & 3x \\
 & \underline{3x + \tfrac{9}{4}x^2} \\
2 + 3x - \tfrac{9}{8}x^2 & -\tfrac{9}{4}x^2 \\
 & \underline{-\tfrac{9}{4}x^2 - \tfrac{27}{8}x^3 + \tfrac{81}{64}x^4} \\
2 + 3x - \tfrac{9}{4}x^2 + \tfrac{27}{16}x^3 & \tfrac{27}{8}x^3 - \tfrac{81}{64}x^4 \\
 & \underline{\tfrac{27}{8}x^3 + \tfrac{81}{16}x^4 - \tfrac{243}{64}x^5 + \tfrac{729}{256}x^6} \\
 & -\tfrac{405}{64}x^4 + \tfrac{243}{64}x^5 - \tfrac{729}{256}x^6
\end{array}$$

EXERCISES.

Extract the square root of the following to four terms:

1. $1 + x$.
2. $1 + 4x + 8x^2$.
3. $1 - x$.
4. $1 + x + x^2$.
5. $1 + 5x + x^2$.
6. $1 - 4x + 3x^2$.
7. $1 - 3x + 8x^2 + x^3$.
8. $1 + x + x^2 + x^3 + x^4$.

125. Square Root of Arithmetical Number. The following principles may be easily verified by the student;

(1) *The square of a number of one digit consists of one or two digits.*

(2) *The square of a number of two digits consists of three or four digits.*

(3) *The square of a number of three digits consists of five or six digits; and so on.*

From these principles we can at once tell how many digits in the square root of a given number. The square root of 390625 will consist of three digits.

The number of digits in the root may be indicated by separating the given number into groups of two figures each, beginning at the right. The left group may contain either *one* or *two* digits. 39 06 25 and 1 93 21.

Now solve by using the same method as in algebraic problems.

Extract the square root of 390625.

	39 06 25	600 + 20 + 5
	36 00 00	
1200, trial divisor	3 06 25	
20	2 44 00	
1220, complete divisor	62 25	
1240, trial divisor	62 25	
5		
1245, complete divisor		

It is usual to put the work in the following shorter form:

	39 06 25	625
	36	
122	3 06	
	2 44	
1245	62 25	
	62 25	

EXERCISES.

Find: 1. $\sqrt{104976}$. 2. $\sqrt{278784}$. 3. $\sqrt{57121}$.
4. $\sqrt{61504}$. 5. $\sqrt{235225}$. 6. $\sqrt{1092025}$.

126. Roots of Decimals. Group both ways from the decimal point, and solve exactly as in whole numbers.

36 74.87 63 64. This shows the root to be made of two whole number places and three decimal places.

EXERCISES.

Find:

1. $\sqrt{18.3184}$.
3. $\sqrt{.133225}$.
5. $\sqrt{1.110916}$.
2. $\sqrt{8.6436}$.
4. $\sqrt{.00300304}$.
6. $\sqrt{2}$.

If the result of Exercise 6 is desired to three decimals, we may write it $\sqrt{2.000000}$, and then proceed as in the above exercises.

7. $\sqrt{5}$ to three decimals.
8. $\sqrt{11}$ to three decimals.

127. Cube Root of Polynomials. The type form is

$$(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3.$$

All polynomials in this form may have their cube roots written by inspection. $x^3 + 6x^2y + 12xy^2 + 8y^3$ may be written, $x^3 + 3x^2(2y) + 3x(2y)^2 + (2y)^3$.

The cube root is readily seen to be $x + 2y$.

By exactly reversing the type form we can extract the cube root of any polynomial which is a perfect cube.

$$\begin{array}{r|l}
 & A^3 + 3A^2B + 3AB^2 + B^3 \;\underline{|\,A + B} \\
 & \underline{A^3} \\
3A^2 + 3AB + B^2 & 3A^2B + 3AB^2 + B^3 \\
 & \underline{3A^2B + 3AB^2 + B^3}
\end{array}$$

The term B of the root is found in $3A^2B$ and is obtained by dividing $3A^2B$ by $3A^2$. This shows $3A^2$ as the trial divisor, and $3A^2 + 3AB + B^2$ as the complete divisor.

Extract cube root of

$$27\,x^6 - 108\,x^5 + 198\,x^4 - 208\,x^3 + 132\,x^2 - 48\,x + 8.$$

SOLUTION.

	$27\,x^6-108\,x^5+198\,x^4-208\,x^3+132\,x^2-48\,x+8 \;\big\lvert\; 3\,x^2-4\,x+2$
	$27\,x^6$
$3(3\,x^2)^2=27\,x^4$, trial divisor	$-108\,x^5+198\,x^4-208\,x^3$
$27\,x^4+3(3\,x^2)(-4\,x)+(4\,x)^2$,	
or $27\,x^4-36\,x^3+16\,x^2$, complete divisor	$-108\,x^5+144\,x^4-\;\;64\,x^3$
$3(3\,x^2-4\,x)^2=27\,x^4-72\,x^3+48\,x^2$, trial divisor	$54\,x^4-144\,x^3+132\,x^2-48\,x+8$
$3(3\,x^2-4\,x)^2+3(3\,x^2-4\,x)2+2^2=27\,x^4-72\,x^3$	
$+66\,x^2-24\,x+4$, complete divisor	$54\,x^4-144\,x^3+132\,x^2-48\,x+8$

128. Rule for Extraction of Cube Root.

(1) *Arrange with respect to some letter.*

(2) *Extract the cube root of the first term for the first term of the root, and subtract the cube from the polynomial.*

(3) *Use three times the square of the root found for a trial divisor, and by dividing the remainder by this divisor get the second term of the root.*

(4) *Add to trial divisor three times the product of the first part of the root and the part of the root last found and the square of the root last found.*

(5) *Subtract the product of the complete divisor and the part of the root last found from the remainder of the polynomial.*

(6) *Repeat this process until the root has been completely determined.*

EXERCISES.

Extract the cube root of:

1. $x^3 + 6x^2 + 12x + 8$.
2. $x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1$.
3. $x^6 - 3x^5y + 6x^4y^2 - 7x^3y^3 + 6x^2y^4 - 3xy^5 + y^6$.
4. $a^3 + 3a^2(b+1) + 3a(b+1)^2 + b^3 + 3b^2 + 3b + 1$.
5. $a^3 + 3a^2b + 6a^2 + 12ab + 3ab^2 + b^3 + 12a + 12b + 6b^2 + 8$.
6. $27x^3 - 64y^3 - 108x^2y + 144xy^2$.
7. $x^6 - 9x^5 + 42x^4 - 117x^3 + 210x^2 - 225x + 125$.
8. $8a^6b^6 - 36a^5b^5 + 78a^4b^4 - 99a^3b^3 + 78a^2b^2 - 36ab + 8$.

129. Extraction of Cube Root of Arithmetical Numbers. The following principles may be verified as in square root:

(1) *The cube of a number of one digit consists of one, two, or three digits.*

(2) *The cube of a number of two digits consists of four, five, or six digits.*

(3) *The cube of a number of three digits consists of seven, eight, or nine digits; and so on.*

From these principles we can at once tell how many digits in the cube root of a given number.

If the given number is separated into groups of three figures each, each group will correspond to a digit of the root.

For example: 95 256 152 263. There are four digits in the cube root of this number. The group at the left may contain one, two, or three digits. 3 365 791 and 871 625.

An example will illustrate the method of solution and show that the same plan is followed as in the extraction of the cube root of polynomials.

Extract the cube root of 262144.

$$\begin{array}{lrr|l}
a^3 + 3\,a^2b + 3\,ab^2 + b^3 & = & 262\,144 & 60 + 4 = 64 \\
a^3 & = 60^3 = & 216\,000 & \\
\hline
(3\,a^2b + 3\,ab + b^2)\,b = & & 46\,144 & \\
3\,a^2 = \quad 3 \times 60^2 = 10800 & & & \\
+\,3\,ab = 3 \times 60 \times 4 = \quad 720 & & & \\
+\,b^2 = 4^2 \qquad = \quad 16 & & & \\
\hline
(3\,a^2b + 3\,ab + b^2)\,b = 11536 \times 4 = 46\,144 & & &
\end{array}$$

The above shows the similarity to the general type.

In practice the solution should appear as follows:

$$\begin{array}{llr|l}
 & & 262\,144 & 64 \\
 & 6^3 = & 216 & \\
\hline
\text{Trial divisor} & 6^2 \times 300 = & 10800 & 46\,144 \\
 & 6 \times 4 \times 30 = & 720 & \\
 & 4^2 = & 16 & \\
\hline
\text{Complete divisor} & & 11536 & 46\,144
\end{array}$$

EXERCISES.

Extract the cube root of each of the following numbers:

1. 4913.
2. 300763.
3. 753571.
4. 614125.
5. 2628072.
6. 1.728.

In Exercise 6, group both ways from the decimal point. If necessary, annex ciphers to fill the last group. If the root is not exact, by annexing ciphers the result may be carried to any desired number of decimal places.

7. 130323.843.
8. 3 to three decimals.
9. .0081 to three decimals.
10. 2.05 to three decimals.

CHAPTER XIII.

THEORY OF INDICES.

130. The Index Laws. The following laws for integral exponents have already been proved:

1. $a^m \times a^n = a^{m+n}$.
2. $a^m \div a^n = a^{m-n}$, when m is greater than n.
3. $(a^m)^n = a^{mn}$.
4. $(a \times b)^m = a^m \times b^m$.
5. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$.
6. $\sqrt[n]{a^m} = a^{\frac{m}{n}}$, when m is divisible by n.

The laws of algebra should be general. Let us assume that the above index laws hold true for all values of m and n, and find consistent meanings for certain special forms, viz., a^0, a^{-n}, $a^{\frac{p}{q}}$.

Any number affected by an exponent is called the base with respect to that exponent.

In a^n, a is the base. In $(x+y)^m$, $x+y$ is the base.

131. The Form a^0. The second index law is

$$a^m \div a^n = a^{m-n}.$$

Let $m = n$, and this becomes

$$a^n \div a^n = a^0.$$

But $$a^n \div a^n = 1.$$

Hence, $$a^0 = a^n \div a^n = 1.$$

Any quantity with an exponent zero is equal to unity.

$$2^0 = 1, \text{ for } 2 \div 2 = 1;\ 10^0 = \frac{10^2}{10^2} = 1;$$

$$(x+y)^0 = \frac{(x+y)^k}{(x+y)^k} = 1;\ \left(\frac{3+a}{5+b}\right)^0 = 1.$$

132. The Form a^{-n}. The first index law states that

$$a^m \times a^n = a^{m+n}.$$

Let us assume that this holds for all values of m and n. Let $m = -n$, and we have

$$a^{-n} \times a^n = a^{-n+n} = a^0 = 1.$$

Hence, $$a^{-n} = \frac{1}{a^n}, \text{ by division.}$$

a^{-n} means the reciprocal of a^n.

$$3^{-5} = \frac{1}{3^5};\ 10^{-1} = \frac{1}{10};\ (x+y)^{-n} = \frac{1}{(x+y)^n};$$

$$ax^{-4} = \frac{a}{x^4};\ ax^{-n}y^{-m} = \frac{a}{x^n y^m};\ 3\,a^{-n} = \frac{3}{a^n}.$$

A factor may be removed from the numerator to the denominator, or vice versa, if at the same time the sign of its exponent be changed.

$$\frac{5^2x^{-3}y^2}{6^{-4}z^{-2}w^4} = \frac{5^2z^2y^2}{6^{-4}x^3w^4} = \frac{5^2 \times 6^4z^2x^{-1}w^{-2}}{x^2y^{-2}w^2} =, \text{ etc.}$$

133. The Form $a^{\frac{p}{q}}$. We know that $(a^m)^n = a^{mn}$.

Put $m = \frac{p}{q}$ and $n = q$. Then $(a^m) = (a^{\frac{p}{q}})^q = a^{\frac{pq}{q}} = a^p$.

Now extract the qth root of both sides of $(a^{\frac{p}{q}})^q = a^p$, and we have $a^{\frac{p}{q}} = \sqrt[q]{a^p}$; that is, $a^{\frac{p}{q}}$ *means the qth root of the pth power of a.*

The numerator of a fractional exponent means the power of the base, and the denominator the root to be extracted.

To extract the qth root of a^p, divide the exponent p by q.

The qth root of a^p is $a^{\frac{p}{q}}$, or $a^{\frac{p}{q}} = \sqrt[q]{a^p}$.

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1.$$

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = (a^{\frac{1}{2}})^2.$$

$$(a^{\frac{1}{2}})^2 = a^1.$$

$$a^{\frac{1}{2}} = \sqrt{a}.$$

$$a^{\frac{3}{4}} = \sqrt[4]{a^3}. \quad a^{\frac{5}{6}} = \sqrt[6]{a^5}.$$

We find that by assuming the generality of the *index laws* we have consistent meanings for *zero*, *negative*, and *fractional* exponents. We will hereafter use the six index laws of page 187 for all values of m and n.

EXERCISES.

1. $\dfrac{3\,a^{-3}}{5\,x^{-2}y} = \dfrac{3\,x^2}{5\,a^3y}.$

2. $\dfrac{5\,x^{\frac{2}{3}} \cdot x^{\frac{3}{4}} \cdot x^{-\frac{5}{12}}}{x^{-\frac{1}{2}} \cdot x^{\frac{1}{3}}} = 5\,x^{\frac{2}{3}+\frac{3}{4}-\frac{5}{12}+\frac{1}{2}-\frac{1}{3}} = 5\,x^{\frac{7}{6}}.$

3. $\dfrac{\sqrt[3]{x^2} \times \sqrt[2]{y^3}}{\sqrt[6]{y^{-3}} \times \sqrt[4]{x^6}} = \dfrac{x^{\frac{2}{3}} \cdot y^{\frac{3}{2}}}{y^{-\frac{1}{2}} \cdot x^{\frac{3}{2}}} = \dfrac{y^{\frac{3}{2}+\frac{1}{2}}}{x^{\frac{3}{2}-\frac{2}{3}}} = \dfrac{y^2}{x^{\frac{5}{6}}}.$

Simplify the following by making all exponents + after combining like numbers:

4. $\dfrac{\sqrt{a}\,\sqrt[3]{b}}{4\sqrt[3]{a}\,\sqrt{b}}.$

5. $\dfrac{5^{\frac{1}{2}} \times 6^{\frac{2}{3}} \times 10^3}{5^{-\frac{1}{2}} \times 6^{+\frac{8}{3}} \times 10^{-\frac{9}{4}}}.$

6. $\dfrac{\sqrt{a^2x^3} \times \sqrt[3]{b^3y^2}}{\sqrt{axby}}.$

7. $\dfrac{3^{\frac{1}{2}}x^{-\frac{1}{2}}y^{\frac{3}{4}}z^{-\frac{5}{6}}}{3^{\frac{1}{2}}x^{-\frac{1}{2}}y^{-\frac{13}{4}}z^{\frac{1}{6}}}.$

8. $\dfrac{\sqrt{a^2x^4}}{\sqrt[3]{b^2x^3}} \times \left(\dfrac{5\,x+y}{3\,x+4\,y}\right)^0.$

9. $(a^{-\frac{2}{3}}b^{\frac{1}{4}}c^{\frac{3}{4}}) \div (a^{-\frac{3}{4}}b^{\frac{1}{8}}c^{-\frac{2}{3}}).$

10. $(a^{-\frac{1}{2}}b^{-m}c^{-p}) \cdot (a^{\frac{1}{2}}b^mc^p) \cdot (a^{\frac{2}{3}}b^{\frac{3}{4}}c^{\frac{1}{5}})^0.$

Remove the parentheses and simplify:

11. $\{[(a^2)^2]^{-2}\}^{\frac{1}{4}}$.

12. $([\{(ax + by)^{\frac{1}{2}}\}^4]^0)^2$.

13. $(x + y^{-1})^2 = x^2 + 2\,xy^{-1} + y^{-2} = x^2 + \frac{2\,x}{y} + \frac{1}{y^2}$.

14. $(x^2 - y^{-2})^2$.

15. $(x^2 - y^{-2})^3$.

It is to be noted that while $(a^{\frac{1}{2}}b^{\frac{1}{2}})^2 = ab$, $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$ does not equal $a + b$. An exponent is *distributive* to the *factors*, but not to the *terms* of an expression.

$$(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 = a + 2\,a^{\frac{1}{2}}b^{\frac{1}{2}} + b.$$

$$(a^2 + b^2)^{\frac{1}{2}} = \sqrt{a^2 + b^2}.$$

Perform the operations indicated in the following:

16. $(x + y^{-1})(x - y^{-1})$.

17. $(x^2 + x + 1 + x^{-1} + x^{-2})(x - x^{-1})$.

18. $(x^2 - y^{-2}) \div (x - y^{-1})$.

19. $(x^2 - 2\,x + x^0 - 2\,x^{-1})(x^2 - x + x^0)$.

20. $(a^{-2} + 2\,a^{-1}b^{\ 1} + b^{-2})(a^{-1} + b^{-1})$.

21. $(a^{\frac{5}{6}}b^{\frac{7}{8}}c^{\frac{1}{3}}x^{\frac{4}{3}})^2 \div (a^{\frac{2}{3}}b^{\frac{3}{4}}c^{\frac{2}{3}}x^{\frac{1}{3}})^3$.

22. $(a^{-\frac{1}{2}}b^{\frac{2}{3}}c^{-\frac{1}{4}}x^{-\frac{1}{3}})^6 \div (a^{\frac{1}{3}}b^{\frac{2}{3}}c^{-1}x^{-2})^{-1}$.

23. $(2\,a^{-3}b^{-2}c^{-\frac{1}{2}} + 3\,a^{-4}b^{-1}c^{\frac{1}{2}}) \div (6\,a^{-3}b^{-2}c^{-1})$.

24. $(9\,x^{\frac{4}{3}}y^{\frac{2}{3}} - 16\,a^{\frac{6}{3}}b^{\frac{2}{3}}) \div (3\,x^{\frac{2}{3}}y^{\frac{1}{3}} + 4\,ab^{\frac{1}{3}})$.

25. $(x^{-2} + x^{-1}y^{-1} + y^{-2})(x^{-2} - x^{-1}y^{-1} + y^{-2})$.

CHAPTER XIV.

RADICALS, SURDS, AND IMAGINARIES.

134. Definitions.

Radicals. A root indicated by means of a *radical* sign is called a *radical.*

As noted in Chapter XII the quantity under the radical sign is called the *radicand.*

$\sqrt{7}$, $\sqrt[3]{5}$, $\sqrt[4]{a^3}$, $\sqrt{x^3+y^3}$ are *radicals*, and 7, 5, a^3, x^3+y^3 are the *radicands.*

These radicals may also be expressed in equivalent expressions by means of fractional exponents. Thus,

$$\sqrt{7} = 7^{\frac{1}{2}},\ \sqrt[3]{5} = 5^{\frac{1}{3}},\ \sqrt[4]{a^3} = a^{\frac{3}{4}},\ \sqrt{x^3+y^3} = (x^3+y^3)^{\frac{1}{2}}.$$

The laws of algebra apply to radicals, since radical signs may be replaced by exponents. All the laws of exponents hold for radicals.

Thus, $$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b},$$

for $$\sqrt[n]{ab} = (ab)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}} = \sqrt[n]{a}\sqrt[n]{b}.$$

This law may be extended to any number of factors.

$$\sqrt[3]{abc} = \sqrt[3]{a}\sqrt[3]{b}\sqrt[3]{c}.$$

Commensurable Numbers. A commensurable number is one whose value may be expressed as a fraction with integral terms.

Thus, $49 = \frac{49}{1} = \frac{98}{2}$, is a commensurable number.

Incommensurable Numbers. A number which can not be expressed as a fraction with integral terms is called an *incommensurable.*

Thus, $\sqrt{2} = 1.4142 \cdots$ is incommensurable.

Surds. A surd is an incommensurable root of a commensurable number.

Thus, $\sqrt{2}$ is a surd, for it is an incommensurable root of a commensurable number. 2 is commensurable, but $\sqrt{2}$ is $1.4142 \cdots$, an incommensurable number.

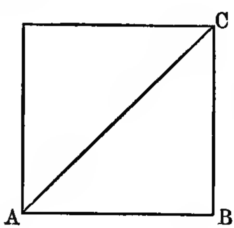
$\sqrt{1+\sqrt{2}}$ is not a surd in the sense of the above definition, for $1 + \sqrt{2} = 2.4142 \cdots$ is itself an incommensurable number.

Examples of Surds: $\sqrt{3}$, $\sqrt[3]{2}$, $\sqrt[4]{5}$, $\sqrt{7}$, $\sqrt{a}$. The latter expression, $\sqrt{a}$, is a surd if a be not a perfect square.

A surd is always a radical, but a radical is not always a surd.

$\sqrt{5}$, $\sqrt[3]{a^2}$, $\sqrt{21}$, $\sqrt{16}$, $\sqrt[3]{x^3y^6}$ are radicals, but $\sqrt{16}$, $\sqrt[3]{x^3y^6}$ are not surds.

135. Surds Expressed Graphically. Many surds may be expressed graphically. In doing this, use is made of the Pythagorean proposition. *In a right triangle the square of the hypotenuse is equal to the sum of the squares of the two legs.*

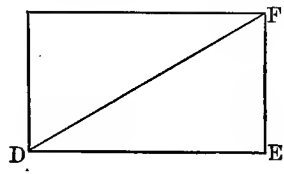


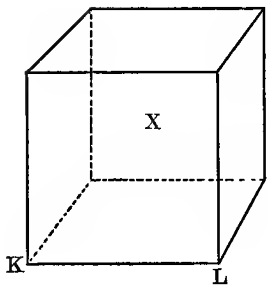
If $\quad AB = BC = 1$,

then $\quad AC = \sqrt{1^2 + 1^2} = \sqrt{2}$.

If $\quad DE = 2$ and $EF = 1$,

then $\quad DF = \sqrt{2^2 + 1^2} = \sqrt{5}$.





If X is a cube whose volume is 5,

then $\quad KL = \sqrt[3]{5}$.

EXERCISES.

Represent graphically

1. $\sqrt{10}$. 2. $\sqrt{3}$. 3. $\sqrt{13}$. 4. $\sqrt{34}$. 5. $\sqrt{6}$.

136. Surd Forms. Radicals whose radicands are algebraic numbers are generally considered *surds* unless the radicand is the nth power of an algebraic number, n being the index of the root.

$\sqrt{a+b}$, $\sqrt[3]{(a+b)^2}$, $\sqrt[4]{a+b+xy}$ are surds.

$\sqrt[3]{(a-x)^3}$, $\sqrt[4]{a^4x^8}$, $\sqrt[2]{x^2+2xy+y^2}$ are radicals, but not surds.

These latter expressions are frequently spoken of as being in the *surd form*.

137. Irrational Numbers. An expression involving a surd or surds is an *irrational*.

$5+\sqrt{6}$, $3+\sqrt{2}-\sqrt{5}$ are *irrationals*.

138. Kinds of Surds. The *order* of a surd is denoted by the index of the root. $\sqrt{5}$, $\sqrt[3]{4}$, $\sqrt[n]{x}$ are surds of second, third, and nth orders respectively. Surds of the second order are generally called *quadratic surds*, those of the third order *cubic surds*, etc.

Surds are of the same order when they have the same root index. $\sqrt[3]{16}$, $\sqrt[3]{x}$, $\sqrt[3]{a+b}$ are surds of the same order.

A *monomial surd* consists of a single surd term.

A *binomial surd* consists of two surd terms, or a surd and a rational term. $\sqrt{5}+\sqrt[3]{2}$, $5+\sqrt{10}$ are binomial surds.

A *trinomial surd* consists of three terms, two of which at least are surds. $3+\sqrt{2}-\sqrt[3]{5}$ and $\sqrt{3}+\sqrt[3]{2}-4\sqrt{5}$ are trinomial surds.

A *mixed surd* consists of a rational factor and a surd factor. $5\sqrt{3}$, $4\sqrt[3]{2}$, $a\sqrt{x+y}$ are mixed surds.

A *surd* is in its *simplest form* when the root index is the smallest possible and the radicand the simplest possible integral expression.

$$\sqrt{27}=\sqrt{9\times 3}=\sqrt{9}\times\sqrt{3}=3\sqrt{3} \text{ (simplest form).}$$

$$\sqrt[4]{36}=\sqrt{6} \text{ (simplest form).}$$

$$\sqrt{\frac{b}{a}}=\sqrt{\frac{ab}{a^2}}=\frac{1}{\sqrt{a^2}}\times\sqrt{ab}=\frac{1}{a}\sqrt{ab}.$$

Similar surds are those which, when reduced to their simplest form, have the same surd factor.

$5\sqrt[3]{4}$, $8\sqrt[3]{4}$ are similar surds.

In this chapter only quadratic surds will be considered.

139. Transformation of Quadratic Surds. A rational quantity may be put in *quadratic surd form* by squaring

it and indicating its square root. Thus, $5 = \sqrt{5^2} = \sqrt{25}$. This is readily extended to the case of reducing a mixed surd to an entire surd.

$$5\sqrt{3} = \sqrt{25 \times 3} = \sqrt{75};\ 4\sqrt{2} = \sqrt{16 \times 2} = \sqrt{32}.$$

A *quadratic surd* is reduced to its simplest form by factoring the radicand and removing to the left of the radical one of each pair of equal factors.

(1) $$\sqrt{72} = \sqrt{2^2 \cdot 3^2 \cdot 2} = 6\sqrt{2}.$$

(2) Simplify $\sqrt{75} + \sqrt{243} - \sqrt{108} - \sqrt{27}$.

$$\sqrt{75} = \sqrt{5^2 \times 3} = 5\sqrt{3}.$$
$$\sqrt{243} = \sqrt{9^2 \times 3} = 9\sqrt{3}.$$
$$\sqrt{108} = \sqrt{6^2 \times 3} = 6\sqrt{3}.$$
$$\sqrt{27} = \sqrt{3^2 \times 3} = 3\sqrt{3}.$$

The expression now is

$$5\sqrt{3} + 9\sqrt{3} - 6\sqrt{3} - 3\sqrt{3} = 5\sqrt{3}.$$

EXERCISES.

Simplify the following surd expressions:

1. $2\sqrt{3} - 4\sqrt{3} + 6\sqrt{3}$.
2. $\sqrt{12} + \sqrt{27} - \sqrt{75}$.
3. $\sqrt{50} - \sqrt{32} + 2\sqrt{18} - 5\sqrt{8}$.
4. $\sqrt{300} + \sqrt{108} - \sqrt{243}$.
5. $3\sqrt{x^3y^4} - 2\sqrt{x^5y^2} + 13\sqrt{xy^6}$.
6. $\sqrt{25\,a^3b^3} - \sqrt{81\,ab} + \sqrt{144\,a^5b^5}$.
7. $\sqrt{3\,(x+y)^2} - 5\sqrt{27\,(x^2 + 2\,xy + y^2)} + \sqrt{12\,(x+y)(x+y)}$.
8. $\sqrt{a^2x^4(y+z)^3} - \sqrt{9\,a^4x^2(y+z)} + 3\sqrt{16\,(y+z)^3}$.

9. $\sqrt{125}+\sqrt{245}-\sqrt{320}+\sqrt{405}-\sqrt{720}.$

10. $\sqrt{8x^3-24x^2+18x}-\sqrt{2x^3-12x^2+18x}.$

11. $\sqrt{8m^3-16m^2n+8mn^2}-\sqrt{2m^3+4m^2n+2mn^2}.$

12. $(x-y)\sqrt{3}+\sqrt{3x^2+6xy+3y^2}-(x+y)\sqrt{108}.$

13. $\sqrt{4x^3+4x^2y}+\sqrt{4xy^2+4y^3}.$

14. $5\sqrt{9a^2b+27a^2}+7a\sqrt{25b+75}.$

15. $\sqrt{48xy^2}+y\sqrt{75x}+\sqrt{3x(x-9y)^2}.$

16. $\sqrt{75a^3}-\sqrt{3a^3+27ab^2-18a^2b}+\sqrt{27ab^2}.$

17. $\sqrt{5(a-b)^2}-\sqrt{20a^2+40ab+20b^2}+\sqrt{20a^2-40ab+20b^2}.$

18. $\sqrt{99}-\sqrt{176}+\sqrt{539}+4\sqrt{275}.$

19. $\sqrt{52}-3\sqrt{117}+5\sqrt{1573}.$

20. $(a+b)\sqrt{(a-b)^2(x+y)}+(a-b)\sqrt{(x+y)(a+b)^2}$
$(3a-4b)\sqrt{(3a+4b)^2(x+y)}.$

140. Product of Quadratic Surds. Products of surds obey the following law:

$$\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}.$$

(1)
$$\begin{aligned}\sqrt{27}\times\sqrt{32}&=3\sqrt{3}\times4\sqrt{2}\\&=12\sqrt{3}\times\sqrt{2}\\&=12\sqrt{6}.\end{aligned}$$

(2) $$\sqrt{3(a+b)}\times\sqrt{2(a-b)}=\sqrt{6(a^2-b^2)}.$$

(3) $$\sqrt{150}\div\sqrt{48}=5\sqrt{6}\div4\sqrt{3}=\tfrac{5}{4}\sqrt{\tfrac{6}{3}}=\tfrac{5}{4}\sqrt{2}.$$

EXERCISES.

Multiply the following surds:

1. $\sqrt{20}\times\sqrt{80}.$

2. $\sqrt{32}\times\sqrt{200}\times\sqrt{50}.$

3. $5\sqrt{3}\times\sqrt{48}\times\sqrt{75}\times\sqrt{15}.$

4. $3\sqrt{ax}\times\sqrt{16a^3x}\times\sqrt{48ax^3}.$

5. $\sqrt{(x-y)^2} \times \sqrt{(x-y)^3}$.

6. $\sqrt{60\,x^3y^3} \times \sqrt{135\,xy} \times \sqrt{36\,x^4y^4}$.

7. $\sqrt{3\,(a-b)^3} \times \sqrt{2\,(a-b)} \times \sqrt{6\,(a-b)^5}$.

8. $\sqrt{5\,(x-y)^2} \times \sqrt{20\,(x+y)^2} \times \sqrt{3\,(x-y)^3(x^2+xy+y^2)}$.

Simplify:

9. $\dfrac{\sqrt{5\,ax^2} \times \sqrt{72\,a^2(x+y)^2} \times \sqrt{2\,(x-y)}}{\sqrt{25\,a^2x^2} \times \sqrt{32\,a\,(x-y)^2(x+y)}}$.

10. $(\sqrt{32}+\sqrt{48}) \div (\sqrt{2}+\sqrt{3})$.

141. Multiplication of Polynomial Surd Expressions. Such expressions as

$$(a+\sqrt{b}+\sqrt{c}) \times (a-\sqrt{b}+\sqrt{c})$$

are multiplied together in the same way as integral expressions, the extended law for multiplication of radicals being observed.

(1)
$$\begin{array}{llll}
a & +\sqrt{b} & +\sqrt{c} & \\
a & -\sqrt{b} & +\sqrt{c} & \\
\hline
a^2 & +a\sqrt{b} & +a\sqrt{c} & \\
 & -a\sqrt{b} & & -b-\sqrt{bc} \\
 & & +a\sqrt{c} & +\sqrt{bc}+c \\
\hline
a^2 & & +2\,a\sqrt{c}-b & +c
\end{array}$$

(2) $(1+\sqrt{3}) \times (\sqrt{2}-\sqrt{5}) = \sqrt{2}+\sqrt{6}-\sqrt{5}-\sqrt{15}$.

EXERCISES.

Multiply the following:

1. $(1+\sqrt{3}) \times (1-\sqrt{3})$.

2. $(2-\sqrt{5}) \times (2+\sqrt{5})$.

3. $(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})$.

4. $(2\sqrt{3}-1)(2\sqrt{3}+1)$.

5. $(3\sqrt{2}-4)(3\sqrt{2}+4)$.

6. $(6\sqrt{3}-2)(3\sqrt{3}-1)$.

7. $(\sqrt{2}+\sqrt{5})^2$.

8. $(\sqrt{2}+\sqrt{3}-1)^2$.

9. $(2\sqrt{3}+1)(\sqrt{3}+2)$.

10. $(\sqrt{5}+\sqrt{3}+4)(\sqrt{5}-\sqrt{3}-4)$.

11. $(2\sqrt{3}-3\sqrt{5}+1)(5\sqrt{3}+2\sqrt{5}-1)$.

12. $(\frac{1}{2}\sqrt{7}-\frac{1}{4}\sqrt{6})(\frac{1}{2}\sqrt{7}+\frac{1}{4}\sqrt{6})$.

13. $(\sqrt{3}+1)(\sqrt{9}-\sqrt{3}+1)$.

14. $(\sqrt{a}+\sqrt{b}-\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c})$.

15. $(2\sqrt{x}-3\sqrt{y}-5\sqrt{z})(2\sqrt{x}+3\sqrt{y}-5\sqrt{z})$.

16. $\dfrac{\sqrt{5}-\sqrt{7}}{2\sqrt{3}-4\sqrt{5}}\times\dfrac{2\sqrt{5}+2\sqrt{7}}{3\sqrt{3}+6\sqrt{5}}$.

142. Conjugate Binomial Surds.

Two quadratic binomial surds differing only in the sign of a surd term are called conjugate surds.

$\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$, or $x+\sqrt{y}$ and $x-\sqrt{y}$,

are *conjugate binomial surds.*

$$(\sqrt{a}+\sqrt{b})\times(\sqrt{a}-\sqrt{b})=\sqrt{a^2}-\sqrt{b^2}=a-b.$$
$$(x+\sqrt{y})\times(x-\sqrt{y})=x^2-\sqrt{y^2}=x^2-y.$$

The product of two conjugate binomial surds is rational.

143. Rationalizing Factors. *When the product of two surd expressions is rational, one expression is said to be the rationalizing factor of the other.*

For example: $\sqrt{a}-\sqrt{b}$ is the rationalizing factor of $\sqrt{a}+\sqrt{b}$, because $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$, a rational result.

$x+\sqrt{y}$ is the rationalizing factor of $x-\sqrt{y}$, because $(x-\sqrt{y})(x+\sqrt{y})=x^2-y$.

$\sqrt{5}$ is the rationalizing factor of $\sqrt{5}$, because $\sqrt{5}\times\sqrt{5}=5$.

Suppose the value of $\frac{\sqrt{5}}{\sqrt{5}-\sqrt{3}}$ is required. It will evidently be very much simpler if we first rationalize the denominator.

$$\frac{\sqrt{5}}{\sqrt{5}-\sqrt{3}}=\frac{\sqrt{5}(\sqrt{5}+\sqrt{3})}{(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})}=\frac{5+\sqrt{15}}{5-3}=\frac{5+\sqrt{15}}{2}.$$

Again, $$\frac{\sqrt{7}-\sqrt{2}}{\sqrt{2}}=\frac{(\sqrt{7}-\sqrt{2})\sqrt{2}}{\sqrt{2}\sqrt{2}}=\frac{\sqrt{14}-2}{2}.$$

EXERCISES.

Rationalize the denominators of the following fractions:

1. $\frac{\sqrt{5}}{\sqrt{2}}$.
2. $\frac{1+\sqrt{2}}{\sqrt{3}}$.
3. $\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{2}}$.
4. $\frac{2}{1+\sqrt{2}}$.
5. $\frac{5}{4-2\sqrt{3}}$.
6. $\frac{\sqrt{2}}{5-2\sqrt{6}}$.
7. $\frac{3\sqrt{6}}{4-3\sqrt{7}}$.
8. $\frac{1+\sqrt{2}}{1-\sqrt{2}}$.
9. $\frac{2+\sqrt{3}}{\sqrt{3}-1}$.
10. $\frac{3+\sqrt{5}}{3-\sqrt{5}}$.
11. $\frac{4+2\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.
12. $\frac{3-4\sqrt{5}}{2\sqrt{3}-\sqrt{2}}$.
13. $\frac{5+8\sqrt{5}}{2\sqrt{5}-\sqrt{3}}$.
14. $\frac{3+2\sqrt{2}+\sqrt{3}}{2\sqrt{5}+3\sqrt{7}}$.
15. $\frac{3+4\sqrt{a}}{\sqrt{a}+\sqrt{b}}$.
16. $\frac{2+3\sqrt{a}+\sqrt{b}}{2\sqrt{a}+4\sqrt{b}}$.
17. $\frac{1+3\sqrt{x-1}}{1-3\sqrt{x-1}}$.
18. $\frac{\sqrt{x}+\sqrt{y}}{(x+y)+\sqrt{2xy}}$.
19. $\frac{x-\sqrt{xy}+y}{\sqrt{x}-\sqrt{y}}$.
20. $\frac{\sqrt{a-2b}+\sqrt{2a-b}}{\sqrt{a-2b}-\sqrt{2a-b}}$.

144. Rationalizing a Trinomial Surd. A trinomial surd expression may be rationalized by two operations.

(1) Rationalize $\sqrt{a}+\sqrt{b}+\sqrt{c}$. First multiply by $\sqrt{a}+\sqrt{b}-\sqrt{c}$. This gives $a+b-c+2\sqrt{ab}$. Now multiply by $a+b-c-2\sqrt{ab}$. This gives $(a+b-c)^2-4ab$, a rational expression. Hence the rationalizing factor of $\sqrt{a}+\sqrt{b}+\sqrt{c}$ is $(\sqrt{a}+\sqrt{b}-\sqrt{c})(a+b-c-2\sqrt{ab})$.

(2) To rationalize $\sqrt{2}+\sqrt{3}+\sqrt{5}$, we multiply by $(\sqrt{2}+\sqrt{3}-\sqrt{5})(2+3-5-2\sqrt{6})$. This will give a product 24.

EXERCISES.

Find the rationalizing factors of the following:

1. $1+\sqrt{2}+\sqrt{3}$.
2. $\sqrt{3}+\sqrt{5}+\sqrt{7}$.
3. $\sqrt{3}+\sqrt{2}+\sqrt{5}$.
4. $\sqrt{5}-\sqrt{2}+1$.
5. $\sqrt{10}-\sqrt{2}+\sqrt{3}$.
6. $\sqrt{a}+\sqrt{b}+c$.
7. $1-2\sqrt{2}+3\sqrt{3}$.
8. $2\sqrt{a}-\sqrt{2b}+3\sqrt{c}$.

145. Rational Numbers and Surds.

THEOREM I. *If $\sqrt{x}$ and $\sqrt{y}$ be surds, then $\sqrt{x}$ can not equal $a+\sqrt{y}$, where a is rational.*

For, assuming $\sqrt{x}=a+\sqrt{y}$, and squaring, we have

$$x=a^2+y+2a\sqrt{y},$$

or

$$\frac{x-a^2-y}{2a}=\sqrt{y}.$$

But the left member of this supposed equality is rational, and therefore can not equal the surd $\sqrt{y}$.

Hence, $$\sqrt{x}\neq a+\sqrt{y}.$$

The sign $\neq$ is read, is not equal.

THEOREM II. *If* $a+\sqrt{x}=b+\sqrt{y}$, *where* a *and* b *are rational, and if* $\sqrt{x}$ *and* $\sqrt{y}$ *are surds, then* $a=b$ *and* $x=y$.

The proof of this theorem is similar to that of Theorem I. We have

$$a-b+\sqrt{x}=\sqrt{y}.$$

Squaring, $(a-b)^2+2(a-b)\sqrt{x}+x=y$.

Transposing, $-x+y-(a-b)^2=2(a-b)\sqrt{x}$.

Here, a rational number, $-x+y-(a-b)^2$, is equal to $2(a-b)$ times a surd. But this can not be true, except in the case $a=b$; but when $a=b$ the original equality says that $x=y$.

146. Square Root of a Binomial Surd. The square root of certain binomial surds may be extracted.

(1) Find the square root of $5+2\sqrt{6}$.

Let $\sqrt{5+2\sqrt{6}}=\sqrt{x}+\sqrt{y}$.

Then $5+2\sqrt{6}=x+y+2\sqrt{xy}$, by squaring.

$x+y=5,\ xy=6$, by Theorem II.

The question now is to find two numbers whose sum is 5, and whose product is 6. These are seen to be 2 and 3.

Hence, $\sqrt{5+2\sqrt{6}}=\sqrt{2}+\sqrt{3}$.

(2) Find the square root of $8-2\sqrt{15}$.

As in (1), let $\sqrt{8-2\sqrt{15}}=\sqrt{x}-\sqrt{y}$.

Then $8-2\sqrt{15}=x+y-2\sqrt{xy}$.

$$x+y=8,\ xy=15,$$

then $x=5,\ y=3$.

Hence, $\sqrt{8-2\sqrt{15}}=\sqrt{5}-\sqrt{3}$.

EXERCISES.

Extract the square root of the following:

1. $10+2\sqrt{21}$.
2. $12-2\sqrt{35}$.
3. $11+2\sqrt{24}$.
4. $7-\sqrt{40}=7-2\sqrt{10}$.
5. $16+2\sqrt{55}$.
6. $a+b-2\sqrt{ab}$.
7. $12-\sqrt{80}$.
8. $13+2\sqrt{42}$.
9. $49+12\sqrt{5}$.
10. $y-2\sqrt{y-1}$.
11. $87-12\sqrt{42}$.
12. $a+b+c+2\sqrt{ac+bc}$.

147. Imaginaries. An imaginary number has already been defined as the even root of a negative number.

We shall have occasion to use only the *square root* of negative numbers.

$$\sqrt{-5},\ \sqrt{-10},\ \sqrt{-16}=4\sqrt{-1},\ \sqrt{-a},$$

are imaginaries.

It is to be noted that every square root of a negative number may be expressed as a real number multiplied by the square root of -1. Thus,

$$\sqrt{-16}=\sqrt{16}\,\sqrt{-1}=4\sqrt{-1};\ \sqrt{-5}=\sqrt{5}\,\sqrt{-1};$$
$$\sqrt{-a^2}=\sqrt{a^2}\,\sqrt{-1}=a\sqrt{-1};\ \sqrt{-a}=\sqrt{a}\,\sqrt{-1}.$$

The $\sqrt{-1}$ is usually denoted by i, and is called the *imaginary* unit.

148. Some Properties of $i=\sqrt{-1}$.

(1) $i=\sqrt{-1}$.

(2) $i^2=(\sqrt{-1})^2=-1$.

(3) $i^3=(\sqrt{-1})^3=i(i^2)=-i$.

(4) $$i^4 = (i^2)^2 = 1.$$

(5) $$i^5 = i(i^4) = i.$$

(6) $$i^6 = i(i^5) = i^2 = -1.$$

(7) $$i^7 = i(i^6) = -i.$$

(8) $$i^8 = (i^4)^2 = 1.$$

(9) $$i^9 = i(i^8) = i;\ \text{etc.}$$

This table shows that the powers of i repeat the values $\sqrt{-1}$, -1, $-\sqrt{-1}$, 1, in cycles of four.

149. Operations with Imaginaries. All the operations possible with surds are also possible with imaginaries. The properties of i must be observed.

(1) $$\begin{aligned}&\sqrt{-36}+\sqrt{-81}+\sqrt{-100}\\ &=\sqrt{36(-1)}+\sqrt{81(-1)}+\sqrt{100(-1)}\\ &=6\sqrt{-1}+9\sqrt{-1}+10\sqrt{-1}\\ &=6i+9i+10i\\ &=25i.\end{aligned}$$

(2) $$\begin{aligned}&\sqrt{-48}+\sqrt{-75}+\sqrt{-243}\\ &=\sqrt{16(-3)}+\sqrt{25(-3)}+\sqrt{81(-3)}\\ &=4i\sqrt{3}+5i\sqrt{3}+9i\sqrt{3}\\ &=18i\sqrt{3}.\end{aligned}$$

(3) $$\begin{aligned}\sqrt{-5}\times\sqrt{-7}&=\sqrt{5}\,i\times\sqrt{7}\,i\\ &=\sqrt{5}\sqrt{7}\,ii\\ &=\sqrt{35}\,i^2\\ &=-\sqrt{35}.\end{aligned}$$

(4) $(\sqrt{-3}+\sqrt{-2})(\sqrt{-3}-\sqrt{-2})$
$$= (i\sqrt{3}+i\sqrt{2})(i\sqrt{3}-i\sqrt{2})$$
$$= i^2(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})$$
$$= i^2(3-2)$$
$$= -1(3-2)$$
$$= -1.$$

(5) $(3+i\sqrt{2})^2 = 9+6i\sqrt{2}+i^2\cdot 2$
$$= 9+6\sqrt{2}\,i-2$$
$$= 7+6\sqrt{2}\,i.$$

(6) $\dfrac{5+\sqrt{3}\,i}{5-\sqrt{3}\,i}.$

To rationalize and make real the denominator, multiply both terms of the fraction by $5+\sqrt{3}\,i$.

$$\frac{(5+\sqrt{3}\,i)}{(5-\sqrt{3}\,i)}\,\frac{(5+\sqrt{3}\,i)}{(5+\sqrt{3}\,i)} = \frac{25+10\sqrt{3}\,i-3}{25-3\,i^2} = \frac{22+10\sqrt{3}\,i}{25-(-3)}$$
$$= \frac{22+10\sqrt{3}\,i}{28}$$
$$= \frac{11+5\sqrt{3}\,i}{14}.$$

EXERCISES.

1. $\sqrt{-18}+\sqrt{-128}-\sqrt{-50}=?$

2. $(3\,i+\sqrt{2}\,i)\times 5\sqrt{2}\,i=?$

3. $(4-\sqrt{3}\,i)^2=?$

4. $(\sqrt{-3}+\sqrt{2})(\sqrt{-2}+3)=?$

5. $\dfrac{5\sqrt{3}}{\sqrt{3}-\sqrt{-2}}=?$

(Make denominator real and rational.)

6. $\dfrac{3+\sqrt{-5}}{3-\sqrt{-5}}=?$

7. $\dfrac{3-2\sqrt{3}\,i}{3+2\sqrt{3}\,i}=?$

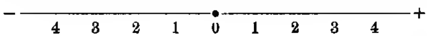
8. $\sqrt{-75}\div\sqrt{-25}=?$

9. $(x\sqrt{-x}+y\sqrt{-y})(x\sqrt{-x}-y\sqrt{-y})=?$

10. $\dfrac{a+bi}{a-bi}=?$ **11.** $\dfrac{\sqrt{-10x^5}}{\sqrt{-5x^3}}=?$

12. $\left(b-\dfrac{1+2i}{2}\right)\left(b-\dfrac{1-2i}{2}\right)=?$

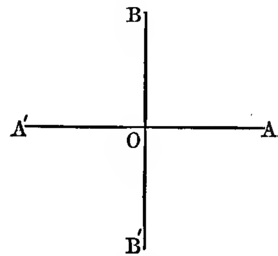
150. Graphical Representation of Imaginaries. Complex Numbers. We are accustomed to represent real numbers upon a straight line, the positive numbers in one direction and the negative numbers in the opposite direction.



Let $OA = 1,$

$OA' = -1,$

$1 \times i^2 = -1.$

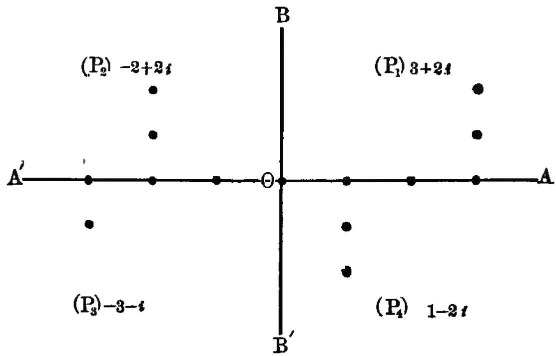


Hence, multiplying 1 by i^2 turns it so as to reverse it in direction. OA is turned about O to the position OA' when it is multiplied by i^2; that is, OA is turned through a half circle. From this we conclude that, when OA is multiplied by i, it will be turned half as far; that is, to the position OB. If OA is multiplied by i^3, it will be turned to the position OB'. It is customary to regard BB' as the line of imaginaries.

A number like $3+2i$ is called a *complex number.* The type form of such a number is

$$a+bi.$$

The following diagram shows how complex numbers are graphically represented.



EXERCISES.

Locate on a diagram the following complex numbers:

1. $4 + i$.
2. $-3 - 3i$.
3. $2 - 5i$.
4. $-3 + i$.
5. $5 - \frac{3}{2}i$.
6. $\frac{3}{2} + \frac{7}{2}i$.

CHAPTER XV.

QUADRATIC EQUATIONS IN A SINGLE VARIABLE.

151. Definition. *An equation in which the highest power of the variable is two is called a quadratic equation.*

Thus, $3x^2 + 4x + 5 = 0$, $x^2 - 5x + 1 = 0$, $ax^2 + b = 0$, are quadratics.

152. Type. Every quadratic in a single variable may be reduced to the type $\boldsymbol{ax^2 + bx + c = 0}$, where $\boldsymbol{a}$ may be any quantity except **0**, $\boldsymbol{b}$ may be any quantity, and $\boldsymbol{c}$ may be any quantity.

$3x^2 - 4x + 5 = 0$ is a special form of the type, in which $\boldsymbol{a} = 3$, $\boldsymbol{b} = -4$, and $\boldsymbol{c} = 5$.

EXERCISES.

Reduce the following equations to the form of the type $\boldsymbol{ax^2 + bx + c = 0}$, and indicate the particular values of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ in each case:

1. $13x^2 = 7x - 5$.
2. $6x = \frac{2}{3} - x^2$.
   (Clear of fractions.)
3. $\dfrac{2x - 7}{3} = \dfrac{2 - x^2}{5}$.
4. $\dfrac{x^2 - 11}{7} = \dfrac{3x - 2}{2}$.
5. $\dfrac{x - 4}{5} = \dfrac{7x^2 - 3}{2}$.
6. $x + \dfrac{1}{x} = 5 - \dfrac{3}{x}$.

7. $\dfrac{x^2-4}{x+1}=2x+\dfrac{3}{4}.$

8. $\dfrac{x^2+5}{x+2}=x+6.$

9. $\dfrac{3x^2+1}{x}=2x.$

10. $\dfrac{3x+1}{x+1}=\dfrac{2x+2}{x+2}.$

153. The Pure Quadratic. If $b=0$, the general quadratic $ax^2+bx+c=0$ becomes $ax^2+c=0$, an equation sometimes called a *pure quadratic*.

The solutions of $ax^2+c=0$ are easily found.

$$\left\{\begin{array}{ll}\text{Transposing } c, & ax^2=-c.\\ \text{Dividing by } a, & x^2=-\dfrac{c}{a}.\\ \text{Extracting the square root, } & x=\pm\sqrt{\dfrac{-c}{a}}.\end{array}\right.$$

If a and c have like signs, the roots $\pm\sqrt{-\dfrac{c}{a}}$ are both *imaginary;* if a and c have unlike signs, the roots are both *real.*

Thus, $3x^2+5=0$ has the roots $x=\pm\sqrt{\dfrac{-5}{3}}$, which are imaginary; $3x^2-5=0$ has the roots $x=\pm\sqrt{\dfrac{5}{3}}$, which are real.

The roots are real and rational if a and c have unlike signs, and $\dfrac{c}{a}$ is a perfect square; the roots are irrational when $\dfrac{c}{a}$ is not a perfect square.

Thus,

$3x^2-5=0$ has the roots $x=\pm\sqrt{\dfrac{5}{3}}$, which are irrational;

$3x^2-27=0$ has the roots $x=\pm\sqrt{\dfrac{27}{3}}=\pm 3$, which are rational.

EXERCISES.

Reduce the following equations to the form of the type:

$$ax^2 + c = 0.$$

1. $\frac{x^2}{7} + \frac{11}{3} = 0.$

   (Clear of fractions.)

2. $\frac{x-3}{5} = \frac{3}{x+3}.$

3. $\frac{x}{7} + \frac{2x^2-5}{14x} = 0.$

4. $\frac{x-3}{x+5} = \frac{5-x}{x+3}.$

5. $\frac{3x^2-7}{7x} = \frac{x}{11}.$

6. $\frac{4x-5}{3x} = \frac{3x}{4x+5}.$

7. $\frac{3x+4}{5x+3} = \frac{5x-3}{3x-4}.$

8. $\frac{4x-5}{x} = \frac{x}{4x+5}.$

9. $\frac{x}{3} + \frac{4}{x} = \frac{5}{x}.$

10. $\frac{3x-7}{x} = \frac{x}{3x+7}.$

EXERCISES.

Solve the following equations:

1. $$7x^2 - 112 = 0.$$

First Solution.

Transposing, $7x^2 = 112.$

Dividing by 7, $x^2 = 16.$

Extracting the square root, $x = \pm 4$, rational roots.

Second Solution.

The general pure quadratic

$$ax^2 + c = 0$$

has the solutions

$$x = \pm\sqrt{-\frac{c}{a}}.$$

Comparing $7x^2 - 112 = 0$ with this general equation,

$$a = 7, c = -112.$$

Hence, $$x = \pm\sqrt{\frac{-(-112)}{7}} = \pm\sqrt{\frac{112}{7}} = \pm\sqrt{16} = \pm 4.$$

2. $\frac{x-3}{6}=\frac{12}{x+3}$.

3. $\frac{x-3}{7}=\frac{x^2-6x+49}{14x}$.

4. $(x+3)(x-3)=2x^2-45$.

5. $3x^2-\frac{2}{3}x^2=21$.

6. $\frac{5x}{x+4}=\frac{x-4}{3x}$.

7. $x^2+\frac{3x-5}{2}=\frac{3}{2}x$.

8. $\frac{x^2}{2}+\frac{3x}{5}+6=\frac{3x+30}{5}$.

9. $-x^2+5+\frac{x+1}{3}=\frac{x}{3}+16x^2$.

10. $-ax^2+25b=0$.

154. Solution of the General Quadratic, $ax^2+bx+c=0$.

Dividing through by a, we have

$$x^2+\frac{b}{a}x+\frac{c}{a}=0.$$

Transposing, $$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

Adding $\frac{b^2}{4a^2}$ (the square of half the coefficient of x) to both sides, we have

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$

$$=\frac{b^2-4ac}{4a^2}.$$

This is called completing the square, because it always makes the first member of the equation a square.

Extracting the square root of both sides,

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}.$$

Transposing, $$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

This result shows that the quadratic $ax^2 + bx + c = 0$ has two roots; namely,

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and $$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

This is the solution of the general quadratic and may be taken as the type solution for all quadratics.

Solve $$3x^2 + 8x + 5 = 0.$$

FIRST SOLUTION.

Dividing by 3, $$x^2 + \tfrac{8}{3}x + \tfrac{5}{3} = 0.$$

Transposing, $$x^2 + \tfrac{8}{3}x = -\tfrac{5}{3}.$$

Completing the square,

$$x^2 + \tfrac{8}{3}x + \tfrac{16}{9} = \tfrac{16}{9} - \tfrac{5}{3} = \tfrac{1}{9}.$$

Extracting the square root,

$$x + \tfrac{4}{3} = \pm\tfrac{1}{3}.$$

$$x = -\tfrac{4}{3} \pm \tfrac{1}{3}$$

$$= -1 \text{ and } -\tfrac{5}{3}.$$

Verifying the Solution: Put $x = -1$ in the equation $3x^2 + 8x + 5 = 0$ and

$$3(-1)^2 + 8(-1) + 5 = 3 - 8 + 5 = 0.$$

Put $x = -\tfrac{5}{3}$ and

$$3(-\tfrac{5}{3})^2 + 8(-\tfrac{5}{3}) + 5 = \tfrac{25}{3} - \tfrac{40}{3} + 5 = \tfrac{25}{3} - \tfrac{40}{3} + \tfrac{15}{3} = 0.$$

Replacing the variable of an equation by a root reduces the equation to an identity. This process is called *verifying the solution.*

Second Solution.

Comparing $3x^2 + 8x + 5 = 0$ with $ax^2 + bx + c = 0$,

$$a = 3,\ b = 8,\ \text{and}\ c = 5.$$

Let us substitute these values in the type solution, viz.,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

and we have
$$x = \frac{-8 \pm \sqrt{64 - 60}}{6}$$
$$= \frac{-8 \pm \sqrt{4}}{6}$$
$$= \frac{-8 \pm 2}{6} = -1 \text{ and } -\tfrac{5}{3}.$$

It will be noticed that the substitution in the type solution gives results identical with those obtained by carrying out all the steps of dividing, transposing, completing the square, extracting the root, and transposing.

EXERCISES.

Solve the following equations by both processes in the order of the above illustration and verify your results :

1. $2x^2 - 5x = 3.$
2. $4x^2 + 7x = 15.$
3. $3x^2 = 19x + 14.$
4. $5x^2 - 12 = 4x.$
5. $6x^2 + x - 15 = 0.$
6. $x^2 = \frac{x}{2} + 18.$
7. $5x + 77 = 12x^2.$
8. $15x^2 - 8 = 37x.$
9. $7x^2 + 7 = 50x.$
10. $11x^2 = 39x + 20.$
11. $6x^2 + 7x - 20 = 0.$
12. $6x^2 - 47x + 77 = 0.$
13. $\frac{3}{8}x^2 + 4x + 10 = 0.$
14. $\frac{3}{15}x^2 + \frac{33}{180}x - \frac{1}{12} = 0.$
15. $\frac{2}{3}x - \frac{x^2}{12} = 1.$
16. $1 + \frac{7}{2}x = -3x^2.$
17. $x^2 + 1 = -2.9x.$

18. $\frac{1}{x-2}-\frac{2}{x+2}=\frac{3}{5}$.

19. $\frac{21}{5x+20}-1=\frac{-3}{20-5x}$.

20. $\frac{8}{x}-1=\frac{12}{x^2}$.

21. $2+x-\frac{80}{x}=0$.

22. $\frac{x}{x-3}-\frac{x-3}{x}+\frac{x}{x+3}-\frac{x+3}{x}=\frac{2}{3}$.

23. $x^2+2bx+c=0$.

24. $x^2-(m+n)x+mn=0$.

25. $(a-b)x^2-bx=a$.

26. $(a^2-b^2)x^2-(a^2+b^2)x+ab=0$.

27. $\frac{1}{b-x}-\frac{1}{b+x}=\frac{x^2-3}{b^2-x^2}$.

28. $\sqrt{x^2-5x}=\frac{-6}{\sqrt{x^2-5x}}$.

29. $\sqrt{x^2-8x+5}=\frac{-10}{\sqrt{x^2-8x+5}}$.

30. $\sqrt{ax^2+cx}=\frac{b}{\sqrt{ax^2+cx}}$.

155. The Double Root.

Solve $$2x^2-20x+50=0.$$

Comparing with the type $ax^2+bx+c=0$, we have

$$a=2,\ b=-20,\ c=50.$$

Hence,
$$x=\frac{20\pm\sqrt{400-400}}{4}$$
$$=\frac{20\pm\sqrt{0}}{4}$$
$$=5\pm 0.$$

In this case the roots are $5+0$ and $5-0$; that is, 5 in each case. Hence, the equation has two equal roots. In the above example 5 is said to be a *double root.* The quantity under the radical, $b^2 - 4\,ac$, is $400 - 400$ or 0. Hence, the condition for a *double root*, or *two coincident roots*, is

$$b^2 - 4\,ac = 0,$$

where a, b, c are the coefficients in the general quadratic

$$ax^2 + bx + c = 0.$$

EXERCISES.

Solve the following equations, noting those which have double (coincident) roots:

1. $x^2 - 4\,x + 4 = 0.$
2. $4\,x^2 + 4\,x + 1 = 0.$
3. $3\,x^2 - 2\,x - 1 = 0.$
4. $4\,x^2 - 12\,x + 9 = 0.$
5. $3\,x^2 + 4\,x + 1 = 0.$
6. $x^2 - \frac{1}{2}\,x + \frac{1}{16} = 0.$
7. $5\,x^2 - 4\,x - 1 = 0.$
8. $25\,x^2 + 30\,x + 9 = 0.$
9. $\frac{1}{64}\,x^2 + \frac{1}{4}\,x + 1 = 0.$
10. $-3\,x^2 + 4\,x + \frac{5}{3} = 0.$

156. Irrational Roots.

Solve $3\,x^2 - 9\,x + 2 = 0.$

Comparing with the type

$$ax^2 + bx + c = 0,$$

we have $a = 3,\ b = -9,\ c = 2.$

Hence, $$x = \frac{9 \pm \sqrt{81 - 24}}{6}$$

$$= \frac{9 \pm \sqrt{57}}{6}.$$

Here it will be noticed that the roots are irrational. The quantity under the radical, $b^2 - 4ac$, is $81 - 24$ or 56, and is not a perfect square. The roots are conjugate surds.

EXERCISES.

Solve the following equations:

1. $4x^2 + x - 1 = 0.$
2. $3x^2 - 7x + 3 = 0.$
3. $2x^2 + 11x - 7 = 0.$
4. $5x^2 - 3x - 3 = 0.$
5. $\dfrac{x^2}{3} - \dfrac{4x}{5} - 1 = 0.$
6. $(x-1)(x-2) = 1.$
7. $8x^2 - 21 = 20x.$
8. $\dfrac{2}{x-2} + \dfrac{1}{3x-1} = \dfrac{3}{x-3}.$
9. $20x - \dfrac{13x^2}{2} - \dfrac{14x}{5} = \dfrac{17x^2}{2} - 25\tfrac{1}{5}.$
10. $9m^4n^4x^2 - n^2 = 6m^3n^3x.$

157. Complex Roots.

Solve $5x^2 - 7x + 3 = 0.$

Comparing with the type

$$ax^2 + bx + c = 0,$$

we have $a = 5,\ b = -7,\ c = 3.$

Hence,

$$x = \frac{7 \pm \sqrt{49 - 60}}{10}$$

$$= \frac{7 \pm \sqrt{-11}}{10}$$

$$= \frac{7 \pm \sqrt{11} \cdot i}{10}.$$

In this case the roots are imaginary. The quantity under the radical, $b^2 - 4ac$, is $49 - 60$ or -11, which is negative and therefore its square root is imaginary. The roots in this case are conjugate complex numbers.

EXERCISES.

Solve the following equations:

1. $x^2 - 5x + 8 = 0.$
2. $2x^2 + 9x + 11 = 0.$
3. $3x^2 - 10x + 9 = 0.$
4. $7x^2 - 11x + 8 = 0.$
5. $-3x^2 + 13x = 20.$
6. $x^2 + .5x + .3 = 0.$
7. $-.3x^2 + .8x - .6 = 0.$
8. $\frac{2}{3}x^2 - \frac{5}{6}x + \frac{1}{2} = 0.$

158. The Discriminant. The solution of the general quadratic $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

From the examples just given we see that the character of the roots is determined by the quantity under the radical. This quantity, $b^2 - 4ac$, is called the *discriminant* of the quadratic.

159. Some Conclusions.

(1) *When* $b^2 - 4ac = a$ *square, the roots are real, rational, and different.*

(2) *When* $b^2 - 4ac = 0$, *the roots are real, rational, and equal.*

(3) *When* $b^2 - 4ac = a$ *positive number not a square, the roots are real and conjugate surds.*

(4) *When* $b^2 - 4ac = a$ *negative number, the roots are conjugate complex numbers.*

(1) $4x^2 - 7x + 3 = 0.$

The discriminant $b^2 - 4ac = 49 - 48 = 1.$

Hence, the roots of this equation are real, rational, and different.

(2) $2x^2 - 4x + 2 = 0$.

The discriminant $b^2 - 4ac = 16 - 16 = 0$.

Hence, the roots of this equation are real, rational, and equal.

(3) $5x^2 + 8x - 2 = 0$.

The discriminant $b^2 - 4ac = 64 + 40 = 104$.

Since 104 is not a square, the roots are conjugate surds.

(4) $7x^2 - 5x + 1 = 0$.

The discriminant $b^2 - 4ac = 25 - 28 = -3$.

Since the square root of -3 is imaginary, the roots are conjugate complex numbers.

EXERCISES.

By means of the discriminant tell what kind of roots belong to each of the following equations:

1. $x^2 - 5x - 9 = 0$.
2. $7x^2 + 3x - 1 = 0$.
3. $5x^2 + 9x + 11 = 0$.
4. $6x^2 - 7x + \frac{5}{4} = 0$.
5. $9x^2 - 13x + 4 = 0$.
6. $ax^2 + 2ax - (a - 4) = 0$.
7. $11x^2 - 3x + \frac{2}{11} = 0$.
8. $x^2 - 3x + \frac{9}{4} = 0$.
9. $4x^2 + 13x + 11 = 0$.
10. $ax^2 + 5x - 1 = 0$.

EXERCISES.

Solve the following quadratics by comparison with the solutions of the type:

1. $3x^2 + 5x - 3 = 0$.
2. $3x^2 - 6x + 1 = 0$.
3. $-2x^2 + 8x + 6 = 0$.
4. $6x^2 + x - 2 = 0$.
5. $24x^2 + 14x - 5 = 0$.
6. $x^2 - .55x - .065 = 0$.
7. $5x^2 - 500 = 0$.
8. $6x^2 + 8x = 0$.
9. $9x^2 - 25x + 50 = 0$.
10. $3x^2 + 21x - 5 = 0$.
11. $7x^2 - 4x + 3 = 0$.
12. $\frac{2}{3}x^2 - 5x + \frac{9}{2} = 0$.

13. $\frac{1}{2}x^2 - 6x - \frac{13}{2} = 0.$

14. $x^2 - \frac{3}{4}x - \frac{1}{4} = 0.$

15. $11x^2 - \frac{2}{3}x + \frac{1}{11} = 0.$

16. $\frac{1}{11}x^2 - 5x + 8 = 0.$

160. Graphical Solution of the Quadratic. The general quadratic

$$ax^2 + bx + c = 0$$

is equivalent to the two equations

$$\begin{cases} y = x^2, \\ ay + bx + c = 0. \end{cases}$$

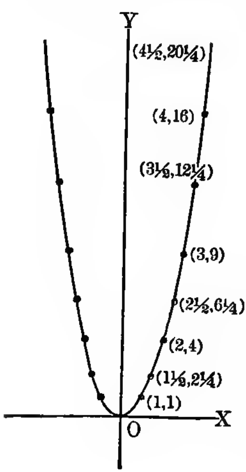
For if in the second of these, we put the value of y from the first, we get the quadratic $ax^2 + bx + c = 0$. We know that $y + bx + c = 0$ has a straight line for its graph. (See page 153.)

Let us see what the graph of $y = x^2$ is.

y can not be negative, because a square can not be negative.

Solving for x, we have

$$x = \pm\sqrt{y}.$$



If $y = 0$, $x = 0$.
$y = 1$, $x = +1$ and -1.
$y = 2$, $x = +1.414$ and -1.414.
$y = 3$, $x = +1.732$ and -1.732.
$y = 4$, $x = +2$ and -2.
etc. etc.

Representing these points with reference to the coördinate axes and drawing a smooth curve through them, we have the curve of the adjacent figure, which is the graph of $y = x^2$.

This curve is a y-parabola and is the same for every quadratic.

The line $ax + by + c = 0$ can only be specifically represented, when we give particular values to a, b, and c.

If the quadratic becomes particular, then the line is specific, and we can readily draw its graph.

Determine by means of graphs the roots of

$$x^2 - x - 2 = 0.$$

This equation is equivalent to

$$\begin{cases} y = x^2, \\ y - x - 2 = 0. \end{cases}$$

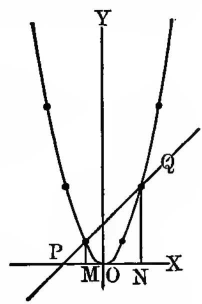
In this $\qquad a = 1;\ b = -1$, and $c = -2$.

In the equation $\qquad y - x - 2 = 0,$

when $y = 0$, $x = -2$, and when $x = 0$, $y = 2$.

Hence, the x-intercept is -2 and the y-intercept is 2.

Now, drawing the graph upon the same diagram that contains the y-parabola, we get the result shown in the adjacent figure.



It is seen that the line cuts the y-parabola at P and Q. The x of P is OM, which is -1, and the x of Q is ON, which is 2.

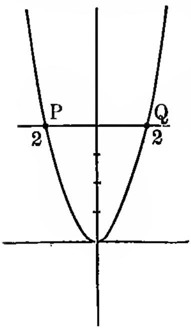
The roots of the quadratic $x^2 - x - 2 = 0$ are -1 and 2.

Hence, we see that the intersections of the line $ax + by + c = 0$ and the y-parabola $y = x^2$ have for their abscissas the roots of the quadratic $ax^2 + bx + c = 0$.

161. Graphical Solution of the Pure Quadratic. If the quadratic is $ax^2 + c = 0$, the two equations to which it is equivalent are

$$\begin{cases} y = x^2, \\ ay + c = 0. \end{cases}$$

Here again we have the same y-parabola. The line $ay+c=0$ is parallel to the x-axis, and so the abscissas of the two points of intersection will be equal in value but opposite in sign.



Determine by means of graphs the roots of

$$3x^2-12=0.$$

The equivalent equations are

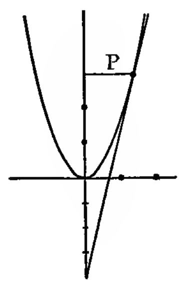
$$\begin{cases} y=x^2, \\ 3y-12=0. \end{cases}$$

The graphs show the two roots to be $+2$ and -2.

162. Graphical Solution in Case of Equal Roots.

Construct graphs showing the roots of

$$x^2-4x+4=0.$$



The equivalent equations are

$$\begin{cases} y=x^2, \\ y-4x+4=0. \end{cases}$$

In this case the line just touches the parabola at P, whose abscissa is 2.

The quadratic $x^2-4x+4=0$ has equal roots, each of them being 2.

163. Graphical Representation in Case of Imaginary Roots.

Construct graphs showing that $x^2-2x+5=0$ has imaginary roots.

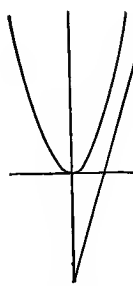
The equivalent equations are

$$y=x^2.$$
$$y-2x+5=0.$$

This equation has imaginary roots. The line

$$y - 2x + 5 = 0$$

does not touch the parabola $y = x^2$.



164. Certain Conclusions.

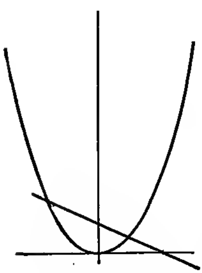
(1) In the case of real different roots, the line $ay + bx + c = 0$ cuts the parabola $y = x^2$ in two places.

(2) In the case of real equal roots, the line $ay + bx + c = 0$ touches the parabola $y = x^2$, but does not cut it.

(3) In the case of imaginary roots, the line $ay + bx + c = 0$ is entirely outside the parabola $y = x^2$.

EXERCISES.

Using the parabola in the book, or a similar one of your own construction, solve by means of graphs the following equations:



1. $2x^2 + x - 1 = 0$.

NOTE. The line in this case is $2y + x - 1 = 0$.

$$x = 0,\ y = \tfrac{1}{2}.$$
$$y = 0,\ x = 1.$$

x and y intercepts are 1 and $\frac{1}{2}$.

Merely lay a ruler across the parabola so as to make these intercepts, and note the abscissas of the points of intersection with the parabola.

2. $x^2 - 3x + 2 = 0$.
3. $4x^2 + 4x + 1 = 0$.
4. $5x^2 - 8x + 3 = 0$.
5. $10x^2 - 3x - 4 = 0$.
6. $5x^2 - 6x - 8 = 0$.
7. $x^2 - 4x + 4 = 0$.
8. $2x^2 - 5x + 3 = 0$.
9. $2x^2 - 4x + 3 = 0$.

165. Relations among the Roots and Coefficients of a Quadratic. The roots of $ax^2 + bx + c = 0$ are

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Here we use α (alpha) and β (beta) to represent the two roots.

Adding these two roots we have

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2b}{2a} = -\frac{b}{a}.$$

Multiplying together these two roots, we have

$$\alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

If the coefficient of x^2 in the general quadratic

$$ax^2 + bx + c = 0$$

be made unity by dividing by a, the equation takes the form

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

The sum of the roots $\alpha + \beta = -\frac{b}{a}$, and the product of the roots $\alpha\beta = \frac{c}{a}$.

Hence, if a quadratic be written so that the coefficient of x^2 is unity, the coefficient of x is the negative of the sum of the roots, and the constant term is the product of the roots.

This has already been seen in factoring, *page* 86.

The roots of the equation $25x^2-15x+2=0$ are $\frac{2}{5}$ and $\frac{1}{5}$. When this equation is divided by 25, it becomes

$$x^2-\tfrac{3}{5}x+\tfrac{2}{25}=0.$$

The sum of the roots is $\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$, and their product is $\frac{2}{25}$. This sum and product are, respectively, the negative coefficient of x and the constant term.

166. Formation of Equations with Given Roots. If the roots of an equation are $\frac{2}{3}$ and 1, what is the equation?

Here $\qquad \alpha=\frac{2}{3},$

and $\qquad \beta=1.$

$$\alpha+\beta=\frac{2}{3}+1=\frac{5}{3}=-\frac{b}{a},$$

$$\alpha\beta=(\tfrac{2}{3})1=\tfrac{2}{3}=\frac{c}{a}.$$

Hence, the equation is

$$x^2-\tfrac{5}{3}x+\tfrac{2}{3}=0,$$

or

$$3x^2-5x+2=0.$$

In general, $x^2-(\alpha+\beta)x+\alpha\beta=0$ is the quadratic equation whose roots are α and β.

EXERCISES.

Make equations which have the following roots:

1. $3,\ -\frac{2}{3}$; $-5,\ 4$; $\frac{2}{3},\ \frac{3}{2}$.
2. $1+\sqrt{5},\ 1-\sqrt{5}$.
3. $3+5i,\ 3-5i$.
4. $\dfrac{2-\sqrt{7}}{3},\ \dfrac{2+\sqrt{7}}{3}$.
5. $5,\ -\frac{11}{3}$.
6. $3-i\sqrt{2},\ 3+i\sqrt{2}$.
7. $3-\frac{5}{12}\sqrt{5},\ 3+\frac{5}{12}\sqrt{5}$.
8. $a+bi,\ a-bi$.
9. $3l+5\sqrt{mi},\ 3l-5\sqrt{mi}$.
10. $\sqrt{5}+\frac{3}{7}\sqrt{3},\ \sqrt{5}-\frac{3}{7}\sqrt{3}$.

167. Generalized Quadratic. $aX^2 + bX + c = 0.$

The above is a quadratic in X. Its roots are

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

X may be any algebraic expression. Whenever an equation may be arranged like the above type, it is said to be of the quadratic form.

(1) $3x^4 + 7x^2 + 4 = 0.$

Put $x^2 = X$, and the equation becomes

$$3X^2 + 7X + 4 = 0,$$

whence $$X = \frac{-7 \pm \sqrt{49 - 48}}{6},$$

$$= -1 \text{ and } -\tfrac{4}{3}.$$

But $$X = x^2,$$

hence, $$x^2 = -1 \text{ and } -\tfrac{4}{3},$$

and $$x = \pm i \text{ and } \pm \tfrac{2}{3}\sqrt{3}\, i.$$

The roots of $$3x^4 + 7x^2 + 4 = 0$$

are $$\pm i, \ \pm \tfrac{2}{3}\sqrt{3}\, i.$$

(2) $$(x^2 - 5)^2 - 7(x^2 - 5) + 12 = 0.$$

Put $x^2 - 5 = X$, and the equation becomes

$$X^2 - 7X + 12 = 0.$$

$$(X - 4)(X - 3) = 0.$$

$$X = 4 \text{ and } 3.$$

Hence, $$x^2 - 5 = 4 \text{ and } 3.$$

$$x^2 = 9 \text{ and } 8.$$

$$x = \pm 3 \text{ and } \pm 2\sqrt{2}.$$

(3) $$2x^2+5\sqrt{x^2-5x+3}=10x+6.$$

Transposing, $2x^2-10x+5\sqrt{x^2-5x+3}-6=0.$

By adding 6 and subtracting 6, we have

$$2x^2-10x+6+5\sqrt{x^2-5x+3}-6-6=0,$$

or, $$2(x^2-5x+3)+5\sqrt{x^2-5x+3}-12=0.$$

Put $\sqrt{x^2-5x+3}=X$, and the equation becomes

$$2X^2+5X-12=0, \quad \text{or} \quad (2X-3)(X+4)=0.$$

$$X=-4 \text{ and } \tfrac{3}{2}.$$

Since $X=\sqrt{x^2-5x+3}$, we have

$$\sqrt{x^2-5x+3}=-4 \text{ and } \tfrac{3}{2}.$$

$$x^2-5x+3=16 \text{ and } \tfrac{9}{4}.$$

$$x^2-5x-13=0 \text{ and } x^2-5x+\tfrac{3}{4}=0.$$

$$x=\frac{5\pm\sqrt{25+52}}{2}=\frac{5\pm\sqrt{77}}{2},$$

and $$x=\frac{5\pm\sqrt{25-3}}{2}=\frac{5\pm\sqrt{22}}{2}.$$

The roots are $\dfrac{5\pm\sqrt{77}}{2}$ and $\dfrac{5\pm\sqrt{22}}{2}$.

EXERCISES.

Solve the following equations:

1. $(x^2+3x)+3\sqrt{x^2+3x}-4=0.$
2. $(x^2-2x)^2-5(x^2-2x)+6=0.$
3. $x+3x^{\frac{1}{2}}+2=0.$
4. $\sqrt{5x^2+4x}+3(5x^2+4x)=24.$
5. $(3x+5)+4\sqrt{3x+5}+7=0.$

6. $\left(x+\frac{1}{x}\right)^2+3\left(x+\frac{1}{x}\right)-5=0.$

7. $3(2x+4)-4\sqrt{2x+4}+1=0.$

8. $x^4+7x^2-18=0.$

9. $x^4+4x^2+4+5(x^2+2)-6=0.$

10. $x^3-5x^{\frac{3}{2}}=36.$

11. $x^4-9x^2=400.$

12. $2x-8+\sqrt{2x-5}=-1.$

13. $x-5-\sqrt{2x-11}=8.$

14. $x^2-2x+2\sqrt{x^2-2x-5}=1.$

15. $x^2+x+\sqrt{x^2+x+5}=25.$

16. $x^2+5x+4=5\sqrt{x^2+5x+28}.$

17. $x^2+5x-\sqrt{x^2+5x+14}=42.$

18. $\frac{3\sqrt{x}+7}{\sqrt{x}+3}=\frac{\sqrt{x}+5}{\sqrt{x}}.$

19. $\sqrt{x^3}-3\sqrt[4]{x^3}=40.$

20. $2x^2-3x+6\sqrt{2x^2-3x+2}=14.$

EXERCISES.

1. One half a number plus the square of the number is 150. Find the number.

2. The sum of two numbers is 15 and their product is 56. Find the numbers.

3. Find two numbers whose sum is 30 and whose product is 216.

4. Separate 41 into two parts such that the product of the part is 330.

5. Two numbers differ by 3, and the sum of their squares is 225. Find the numbers.

6. The sum of the squares of three consecutive numbers is 434. Find the numbers.

7. A rectangular lot is 32 feet longer than it is wide. It contains 13200 square feet. What are the dimensions?

8. A man buys a certain number of chickens for $6. If he had paid 10 cents apiece more for each, he would have gotten 5 fewer for his money. How many chickens did he buy?

9. Find a number such that if its square be diminished by 1, $\frac{3}{4}$ of the remainder is 18 more than 10 times the number.

10. There are two numbers whose difference is 8. If 540 is divided by each of these numbers, the difference of the quotients is 18. Find the numbers.

11. One side of a rectangle is 7 feet longer than the other and its diagonal is 13 feet. Find the area.

12. The difference of the reciprocals of two consecutive numbers is $\frac{1}{1560}$. Find the numbers.

13. The difference between the reciprocals, two consecutive odd numbers, is $\frac{2}{483}$. Find the numbers.

14. By increasing his speed 1 mile an hour a man finds that he takes 3 hours less than usual to walk 60 miles. What is his ordinary rate?

15. The larger of two pipes will fill a cistern in 6 minutes less time than the smaller. When both pipes are open, the cistern is filled in $13\frac{1}{3}$ minutes. Find the time required by each pipe to fill the cistern.

16. A and B have a distance of 150 miles to travel. B starts 10 hours before A and arrives 10 hours after A. A travels 2 miles an hour faster than B. What is the rate of each per hour?

17. A rectangular field is 4 times as long as it is wide. If the width is increased 20 rods, its area is doubled. Find the area of the field.

18. What number increased by 4 and squared is equal $\frac{1}{2}$ of itself increased by 10 and squared?

19. A man sold a horse for $144, thereby gaining as many per cent as the horse cost him dollars. What was the cost of the horse?

20. A boat's crew can row 9 miles down a river and back in 4 hours. The rate of rowing in still water is double the rate of the current. Find the rate of rowing and the rate of the current.

21. The hypotenuse of a right-angled triangle is 5 feet longer than the base and 10 feet longer than the perpendicular. Find the sides of the triangle.

22. The sum of two numbers is a and their product is b. What are the numbers?

23. The perimeter of a rectangular field is 168 rds. and its area is 9 A. Find the length of the sides.

24. Two men start at the same time from the vertex of a right angle and walk along its sides at the rate of 3 and 4 miles per hour, respectively. In how many hours are they 50 miles apart?

CHAPTER XVI.

SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS.

168. When simultaneous equations involve quadratics, they must be solved by methods which depend upon the form of the equations. The various methods are shown under the following cases:

Case I.

169. A Linear and a Quadratic. A simultaneous set of this kind may always be solved. The equations are of the forms,

$$\left.\begin{array}{ll}(1) & ax^2 + bxy + cy^2 = d, \\ (2) & lx + my = k.\end{array}\right\}$$

$$(3) \qquad x = \frac{k - my}{l}, \qquad \text{from (2).}$$

$$(4) \quad a\left(\frac{k - my}{l}\right)^2 + b\left(\frac{k - my}{l}\right)y + cy^2 = d,$$

by substituting in (1).

$$(5) \quad (am^2 - blm + cl^2)y^2 + (blk - 2\,akm)y + ak^2 - dl^2 = 0,$$

by rearranging (4).

Equation (5) is a quadratic in y, and therefore has two roots. The substitution of these roots in (3) will give two values of x. Hence, the set of equations has two roots, and only two.

In the above we have used general equations and we have found that the solution depends upon a quadratic in one variable. Such a quadratic can always be solved. Hence the simultaneous set can always be solved.

I. Solve the following equations:

(1) $\left.\begin{array}{r} x^2 + y^2 = 25, \\ 7y - x = 25. \end{array}\right\}$ (2)

(3) $x = 7y - 25$, from (2).

(4) $49y^2 - 350y + 625 + y^2 = 25$, by substituting in (1).

(5) $50y^2 - 350y + 600 = 0$.

(6) $y^2 - 7y + 12 = 0$, by dividing (5) by 50.

(7) $(y - 3)(y - 4) = 0$, by factoring (6).

(8) $y = 3$ and 4.

(9) $x = 7 \times 3 - 25$ and $7 \times 4 - 25$, by substituting in (3).

(10) $x = -4$ and 3.

The roots are $(-4, 3)$ and $(3, 4)$.

Care should be given to the proper association of the values of x and y. It should be remembered that a root is a properly associated value of x and of y.

The graphs of $x^2 + y^2 = 25$ and $7y - x = 25$.

From $x^2 + y^2 = 25$ we have

$$y = \pm\sqrt{25 - x^2}.$$

When $x = 0$, $y = +5$ and -5;

$(0, 5)$, $(0, -5)$ are roots.

When $x = +1$ and -1, $y = 2\sqrt{6}$ and $-2\sqrt{6}$;

$(1, 2\sqrt{6})$, $(1, -2\sqrt{6})$, $(-1, 2\sqrt{6})$, $(-1, -2\sqrt{6})$ are roots.

When $x=2$ and -2, $y=\sqrt{21}$ and $-\sqrt{21}$;

$(2, \sqrt{21})$, $(2, -\sqrt{21})$, $(-2, \sqrt{21})$, $(-2, -\sqrt{21})$ are roots.

When $x=3$ and -3, $y=4$ and -4;

$(3, 4)$, $(3, -4)$, $(-3, 4)$, $(-3, -4)$ are roots.

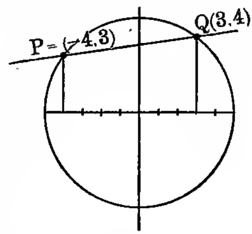
When $x=4$ and -4, $y=3$ and -3;

$(4, 3)$, $(4, -3)$, $(-4, 3)$, $(-4, -3)$ are roots.

When $x=5$ and -5, $y=0$ and 0;

$(5, 0)$ and $(-5, 0)$ are roots.

Locating these roots and drawing a curve through them, we find the graph to be a circle.



The graph of $7y-x=25$ is a straight line. It is the line PQ. This line cuts the circle in the two points P and Q. The coördinates of the two points P and Q, where the line cuts the circle, are $(-4, 3)$ and $(3, 4)$. These are the two roots of the given equations. That they should be the roots appears from the fact that they are the only two points whose coördinates are the same for the line and the circle.

II. Solve the following equations:

$$\left.\begin{array}{lr}(1) & x^2+y^2=25. \\ (2) & 3x+4y=25.\end{array}\right\}$$

(3) $$x=\frac{25-4y}{3}, \qquad \text{from (2).}$$

(4) $$\frac{625-200y+16y^2}{9}+y^2=25, \text{ by substituting in (1).}$$

(5) $$625-200y+16y^2+9y^2=225.$$

(6) $$25y^2-200y+400=0.$$

(7) $$y^2 - 8y + 16 = 0.$$

(8) $$(y - 4)(y - 4) = 0.$$

(9) $$y = 4 \text{ and } 4.$$

(10) $$x = \frac{25 - 4 \times 4}{3} \text{ and } x = \frac{25 - 4 \times 4}{3}.$$

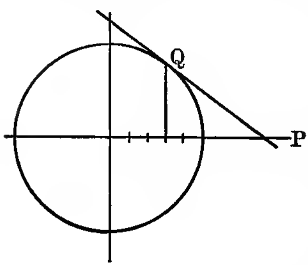
(11) $$x = 3 \text{ and } 3.$$

The roots are (3, 4) and (3, 4).

These roots are the same. Equations (1) and (2) are said to have a *double root*.

The graphs of $x^2 + y^2 = 25$ and $3x + 4y = 25$.

$x^2 + y^2 = 25$ is the same circle as in (1). $3x + 4y = 25$ is a straight line. The graphs are shown in the adjacent figure.



In this case the line PQ, which is the graph of $3x + 4y = 25$, does not cut the circle, but just touches it at the point Q. The coördinates of the point Q are (3, 4). (3, 4) is one of the two equal roots of the given equations. In the case of equal roots the graph of the linear equation just touches the graph of the quadratic equation.

III. Solve the following equations:

$$\left.\begin{aligned} &(1) \quad x^2 + y^2 = 25. \\ &(2) \quad x + y = 10. \end{aligned}\right\}$$

(3) $$x = 10 - y.$$

(4) $$100 - 20y + y^2 + y^2 = 25.$$

(5) $$2y^2 - 20y + 75 = 0.$$

(6) $$y = \frac{20 \pm \sqrt{400 - 600}}{4}$$

$$= \frac{20 \pm 10\sqrt{-2}}{4}$$

$$= \frac{10 \pm 5\sqrt{-2}}{2} = \frac{10 \pm 5\sqrt{2}\,i}{2}.$$

(7) $$x = 10 - \frac{10 \pm 5\sqrt{2}\,i}{2}$$

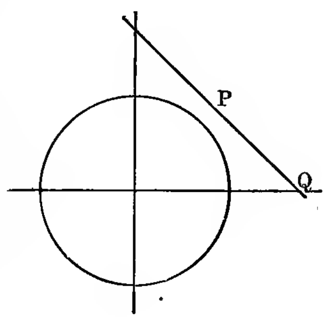
$$= \frac{10 \mp 5\sqrt{2}\,i}{2}.$$

The roots are

$$\left(\frac{10 - 5\sqrt{2}\,i}{2}, \frac{10 + 5\sqrt{2}\,i}{2}\right) \text{ and } \left(\frac{10 + 5\sqrt{2}\,i}{2}, \frac{10 - 5\sqrt{2}\,i}{2}\right).$$

These roots are both *imaginary*.

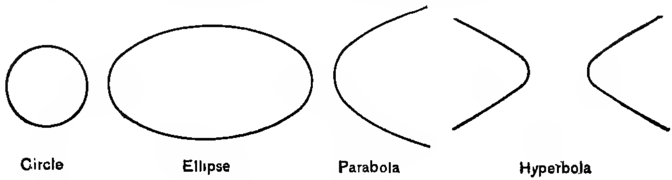
The graphs of $x^2 + y^2 = 25$ and $x + y = 10$ are shown in the adjacent figure.



In this case the line PQ, which is the graph of $x + y = 10$, neither cuts nor touches the circle. In the case of imaginary roots the graph of the linear equation neither cuts nor touches the graph of the quadratic.

170. Graph of the Quadratic in x and y. From the preceding discussion it must not be inferred that the graph

of the quadratic equation in x and y is always a circle. It may be any one of the following curves:



Graph of $$4x^2 + 9y^2 = 36.$$

From this $$y = \sqrt{\frac{36 - 4x^2}{9}}.$$

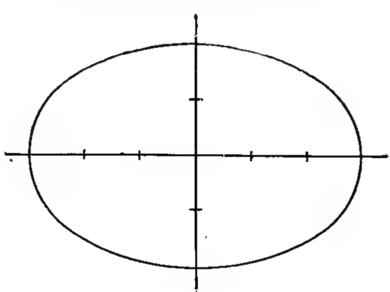
When $x = 0$, $y = 2$ and -2; $(0, 2)$, $(0, -2)$ are roots.

When $x = 1$ and -1, $y = \frac{4}{3}\sqrt{2}$ and $-\frac{4}{3}\sqrt{2}$; $(1, \frac{4}{3}\sqrt{2})$, $(1, -\frac{4}{3}\sqrt{2})$, $(-1, \frac{4}{3}\sqrt{2})$, $(-1, -\frac{4}{3}\sqrt{2})$ are roots.

When $x = 2$ and -2, $y = \frac{2}{3}\sqrt{5}$ and $-\frac{2}{3}\sqrt{5}$; $(2, \frac{2}{3}\sqrt{5})$, $(2, -\frac{2}{3}\sqrt{5})$, $(-2, \frac{2}{3}\sqrt{5})$, $(-2, -\frac{2}{3}\sqrt{5})$ are roots.

When $x = 3$ and -3, $y = 0$; $(3, 0)$, $(-3, 0)$ are roots.

Locating these points and drawing a curve through them, we have an ellipse like the following.



EXERCISES.

Solve the following sets of equations:

1. $\begin{cases} x+y=5, \\ x^2+y^2=25. \end{cases}$

2. $\begin{cases} 4x^2+9y^2=36, \\ 2x+3y=6. \end{cases}$

3. $\begin{cases} x^2+y^2=100, \\ x-y=2. \end{cases}$

4. $\begin{cases} x+y=15, \\ xy=56. \end{cases}$

5. $\begin{cases} x-y=6, \\ xy=27. \end{cases}$

6. $\begin{cases} x-y=4, \\ x^2+y^2=106. \end{cases}$

7. $\begin{cases} 3x+y=9, \\ xy=6. \end{cases}$

8. $\begin{cases} 2x+5=5, \\ 5x^2-xy=2. \end{cases}$

9. $\begin{cases} 3x-2y=8, \\ 4xy=32. \end{cases}$

10. $\begin{cases} x+2y=9, \\ x^2-y^2=21. \end{cases}$

11. $\begin{cases} x^2+3y^2=27, \\ x+y=10. \end{cases}$

12. $\begin{cases} x^2+y^2=26, \\ x+5y=26. \end{cases}$

13. $\begin{cases} 3x-y=12, \\ x^2-y^2=16. \end{cases}$

14. $\begin{cases} 3x+5=4, \\ xy+y^2=9. \end{cases}$

15. $\begin{cases} x+2y=7, \\ \dfrac{3}{x}+\dfrac{10}{y}=1. \end{cases}$

Construct graphs for Exercises 1, 2, 3, 5, 13.

CASE II.

171. Both Equations Quadratic of the Form $ax^2+by^2=c$. When the equations are of this form, one of the variables may be eliminated as in simultaneous equations of the first degree, and the resulting equation is a pure quadratic in the other variable.

I. Solve

$$\begin{cases} (1) & x^2 + y^2 = 16, \\ (2) & 4\,x^2 + 25\,y^2 = 100. \end{cases}$$

Multiplying (1) by 4 and subtracting from (2),

(3) $21\,y^2 = 36.$

(4) $y^2 = \frac{36}{21} = \frac{12}{7}.$

(5) $y^2 = \pm\sqrt{\frac{12}{7}} = \pm\,2\sqrt{\frac{3}{7}}.$

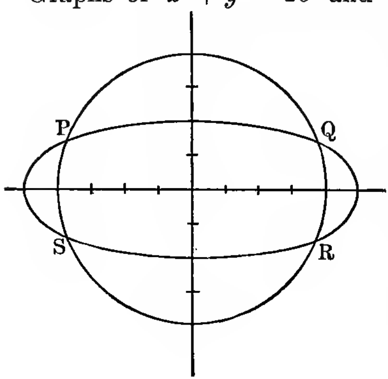
(6) $x^2 + (\pm\sqrt{\frac{12}{7}})^2 = 16,$ substituting in (1).

(7) $x^2 = 16 - \frac{12}{7} = \frac{100}{7}.$

(8) $x = \pm\dfrac{10}{\sqrt{7}}.$

The roots are $\left(-\dfrac{10}{\sqrt{7}},\ 2\sqrt{\dfrac{3}{7}}\right)$, $\left(\dfrac{10}{\sqrt{7}},\ 2\sqrt{\dfrac{3}{7}}\right)$, $\left(\dfrac{10}{\sqrt{7}},\ -2\sqrt{\dfrac{3}{7}}\right)$, $\left(-\dfrac{10}{\sqrt{7}},\ -2\sqrt{\dfrac{3}{7}}\right)$. There are, as in all solutions under this case, four roots.

Graphs of $x^2 + y^2 = 16$ and $4\,x^2 + 25\,y^2 = 100$. The graph of the first equation is a circle and of the second an ellipse. They are shown in the figure.



The graphs of the two equations intersect in the four points P, Q, R, and S. The coördinates of these four points are the four roots of the set of equations.

II. Solve $\begin{cases} (1) & x^2 + y^2 = 25, \\ (2) & 4x^2 + 25y^2 = 100. \end{cases}$

Eliminating x^2 as in Example I,

(3) $$21y^2 = 0.$$

(4) $$y^2 = 0.$$

(5) $$y = 0 \text{ and } 0.$$

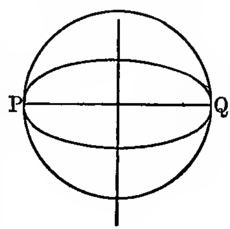
(6) $$x^2 + 0^2 = 25.$$

(7) $$x = \pm 5.$$

The roots are $(5, 0)$, $(5, 0)$, $(-5, 0)$, $(-5, 0)$. The set of equations has two pairs of double roots.

Graphs of $x^2 + y^2 = 25$ and $4x^2 + 25y^2 = 100$. The graphs are a circle and ellipse. They are shown in the adjacent figure.

The graphs of the two equations do not intersect, but they touch each other at the points Q and P. The coördinates of these points are the roots of the equations. As in Case I, when the graphs touch, the coördinates of the points where they touch are double roots.



III. Solve $\begin{cases} (1) & x^2 + y^2 = 1, \\ (2) & 4x^2 + 25y^2 = 100. \end{cases}$

Eliminating x^2 as in Example I,

(3) $$21y^2 = 96.$$

(4) $$y^2 = \tfrac{96}{21}.$$

(5) $$y = \pm\sqrt{\tfrac{96}{21}} = \pm 4\sqrt{\tfrac{2}{7}}.$$

(6) $$x^2 + (\pm\sqrt{\tfrac{96}{21}})^2 = 1.$$

$$x^2 = 1 - \tfrac{96}{21} = -\tfrac{75}{21}.$$

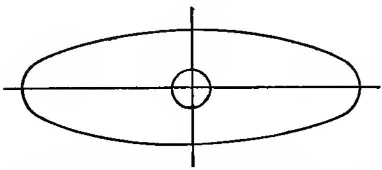
(7) $$x = \pm 5\sqrt{\tfrac{1}{7}}\,i.$$

The roots are $(5\sqrt{\tfrac{1}{7}}\,i,\ 4\sqrt{\tfrac{2}{7}})$, $(5\sqrt{\tfrac{1}{7}}\,i,\ -4\sqrt{\tfrac{2}{7}})$,

$(-5\sqrt{\tfrac{1}{7}}\,i,\ 4\sqrt{\tfrac{2}{7}})$, $(-5\sqrt{\tfrac{1}{7}}\,i,\ -4\sqrt{\tfrac{2}{7}})$.

These roots are all imaginary.

Graphs of $x^2 + y^2 = 1$ and $4x^2 + 25y^2 = 100$ are a circle and an ellipse, respectively. They are shown in the figure.



The graphs of the two equations neither intersect nor touch. The circle is entirely within the ellipse. As in Case I, the roots are imaginary.

EXERCISES.

Solve the following sets of equations: —

1. $\begin{cases} x^2 + y^2 = 13, \\ 2x^2 + y^2 = 17. \end{cases}$

2. $\begin{cases} 2x^2 - y^2 = 2, \\ x^2 + 2y^2 = 41. \end{cases}$

3. $\begin{cases} x^2 + y^2 = 25, \\ 3x^2 + 4y^2 = 180. \end{cases}$

4. $\begin{cases} 6x^2 - y^2 = 5, \\ x^2 + 2y^2 = 107. \end{cases}$

5. $\begin{cases} x^2 + 3y^2 = 52, \\ 2x^2 + 5y^2 = 19. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 34, \\ x^2 - y^2 = 16. \end{cases}$

7. $\begin{cases} x^2 - 9y^2 = 0, \\ 3x^2 + 4y^2 = 12. \end{cases}$

8. $\begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1, \\ \dfrac{x^2}{9} - \dfrac{y^2}{4} = 1. \end{cases}$

Construct graphs for Exercises 1, 6, 8.

Case III.

172. Both Equations Homogeneous in the Part involving the Variables. The equations are of the form

$$ax^2 + bxy + cy^2 = k.$$

The first step of the solution is the elimination of the constant terms of the two equations.

I. Solve $\begin{cases} (1)\ 2x^2 - 3xy + y^2 = 2, \\ (2)\ 2x^2 - 3y^2 = 3. \end{cases}$

Multiply (1) by 3, (2) by 2, and subtract.

(3) $2x^2 - 9xy + 9y^2 = 0.$

(4) $(2x - 3y)(x - 3y) = 0.$

(5) $2x = 3y.$

(6) $x = \frac{3}{2}y.$

(7) $x = 3y.$

Substituting $x = \frac{3}{2}y$ in (2),

(8) $2\left(\frac{3y}{2}\right)^2 - 3y^2 = 3.$

(9) $\frac{9y^2}{2} - 3y^2 = 3.$

(10) $\frac{3y^2}{2} = 3.$

(11) $y^2 = 2.$

(12) $y = \pm\sqrt{2}.$

(13) $x = \frac{3}{2}(\pm 2) = \pm\frac{3}{2}\sqrt{2}.$

Substituting $x = 3y$ in (2),

$$18y^2 - 3y^2 = 3.$$
$$15y^2 = 3.$$
$$y^2 = \tfrac{1}{5}.$$
$$y = \pm\tfrac{1}{5}\sqrt{5}.$$
$$x = 3(\pm\tfrac{1}{5}\sqrt{5}) = \pm\tfrac{3}{5}\sqrt{5}.$$

The roots are

$(\frac{3}{2}\sqrt{2}, \sqrt{2}), (-\frac{3}{2}\sqrt{2}, -\sqrt{2}), (\frac{3}{5}\sqrt{5}, \frac{1}{5}\sqrt{5}), (-\frac{3}{5}\sqrt{5}, -\frac{1}{5}\sqrt{5})$.

In this case there will always be four roots. There may be *one* or *two* pairs of double roots. *Two* or *four* of the roots may be imaginary.

In this case the equations may be of the form

$$ax^2 + bxy + cy^2 = dx \text{ or } ey.$$

They are solved precisely as the above.

II. Solve $\begin{cases} (1)\quad 2x^2 - 3xy + y^2 = 2y, \\ (2)\quad 2x^2 - 3y^2 = 3y. \end{cases}$

Eliminate the right-hand members by multiplying (1) by 3, (2) by 2, and subtracting,

(3) $2x^2 - 9xy + 9y^2 = 0.$

(4) $(2x - 3y)(x - 3y) = 0.$

(5) $x = \frac{3}{2}y$ and $3y$.

Substituting $x = \frac{3}{2}y$ in (2),

(6) $2(\frac{3}{2}y)^2 - 3y^2 = 3y.$

(7) $\frac{3y^2}{2} = 3y,\ y^2 - 2y = 0,\ y(y-2) = 0.$

(8) $y = 0$ and 2.

(9) $x = 0$ and 3.

Substituting $x = 3y$ in (2),

(10) $9y^2 - 3y^2 = 3y.$

(11) $6y^2 = 3y.$

(12) $y(2y - 1) = 0.$

(13) $y = 0$ and $\frac{1}{2}$.

(14) $x = 0$ and $\frac{3}{2}$.

The roots are $(0, 0)$, $(0, 0)$), $(3, 2)$, $(\frac{3}{2}, \frac{1}{2})$. $(0, 0)$ is a double root.

In each of these examples, after finding the value of x in terms of y, the substitution might have been made in the first equation instead of the second. The second was selected because it was of simpler form than the first.

EXERCISES.

Solve the following sets of equations:

1. $\begin{cases} 2x^2 - 3xy + 2y^2 = 4, \\ x^2 + y^2 = 5. \end{cases}$

2. $\begin{cases} x^2 + 2xy - y^2 = 7, \\ 2x^2 - 3xy - y^2 = -1. \end{cases}$

3. $\begin{cases} x^2 + xy = 21, \\ 2xy - y^2 = 8. \end{cases}$

4. $\begin{cases} x^2 - y^2 = 3, \\ x^2 - 2xy + 2y^2 = 2. \end{cases}$

5. $\begin{cases} x^2 - xy + y^2 = 21, \\ 2xy - y^2 = 15. \end{cases}$

6. $\begin{cases} x^2 + 2xy + 2y^2 = 17, \\ 3x^2 - 9xy - y^2 = 119. \end{cases}$

7. $\begin{cases} 2x^2 - xy + y^2 = 16, \\ x^2 + xy + 2y^2 = 44. \end{cases}$

8. $\begin{cases} x^2 + xy = 40, \\ y^2 + xy = 60. \end{cases}$

9. $\begin{cases} 3y^2 - 5x^2 = 70, \\ y^2 - 3xy = 10. \end{cases}$

10. $\begin{cases} lx^2 + my^2 = n, \\ ax^2 + by^2 = c. \end{cases}$

11. $\begin{cases} 4x^2 + 4y^2 = 13 - 4xy, \\ 8x^2 - 12xy = 11 - 8y^2. \end{cases}$

12. $\begin{cases} \dfrac{x^2}{2} + \dfrac{xy}{2} = 37 - y^2, \\ (x + y)^2 = 73 - x^2. \end{cases}$

CASE IV.

173. When Both Equations are Symmetrical in *x* and *y*.

Equations are symmetrical in x and y when the interchange of these letters does not change the equations.

Examples: (1) $x^3 - 3xy + y^3 = 27$. Interchange x and y, and we have $y^3 - 3yx + x^3 = 27$, which is the same as (1).

(2) $x^2 - 2xy + y^2 = 16$,

(3) $x^4 + y^4 = 12$,

(4) $xy = 15$,

and (5) $x + y = 6$,

are all symmetrical equations.

I. Solve $\begin{cases} (1) & x^2 + y^2 = 25, \\ (2) & xy = 12. \end{cases}$

Add $2xy = 24$ to (1), and we have

(3) $x^2 + 2xy + y^2 = 49$,

(4) $x + y = \pm 7$, from (3).

Subtract $2xy = 24$ from (1), and we have

(5) $x^2 - 2xy + y^2 = 1$,

(6) $x - y = \pm 1$.

From (4) and (6) we have, by adding and subtracting,

$$x = 4,\ 3,\ -4,\ -3,$$
$$y = 3,\ 4,\ -3,\ -4.$$

The roots are $(4, 3)$, $(3, 4)$, $(-4, -3)$, $(-3, -4)$.

When both equations are general quadratics, the solution depends upon a cubic or quartic. The investigation of such equations is beyond the compass of this book.

II. Solve $\begin{cases} (1) & x^2 + 2xy - y^2 = 7, \\ (2) & x^2 - 2y^2 + y = 2. \end{cases}$

Solving (2) for x,

(3) $x = \pm\sqrt{2 + 2y^2 - y}$.

Substituting in (1),

(4) $$2 + 2y^2 - y \pm 2y\sqrt{2 + 2y^2 - y} - y^2 = 7.$$

(5) $$\pm 2y\sqrt{2 + 2y^2 - y} = 5 - y^2 + y.$$

Squaring,

(6) $$8y^2 + 8y^4 - 4y^3 = 25 + y^4 - 9y^2 - 2y^3 + 10y.$$

(7) $$7y^4 - 2y^3 + 17y^2 - 10y = 25.$$

This equation is a quartic in y, and unless it breaks up into factors of degree not higher than two it can not be solved by our present methods.

Graphs of $x^2 + y^2 = 25$ and $xy = 12$.

The graph of $x^2 + y^2 = 25$ is a circle.

$$xy = 12.$$

$$y = \frac{12}{x}.$$

When $x = 0$, $y = \infty$; $(0, \infty)$ is a root.

When $x = +1$ and -1, $y = 12$ and -12;
$(1, 12)$, $(-1, -12)$ are roots.

When $x = +2$ and -2, $y = 6$ and -6;
$(2, 6)$, $(-2, -6)$ are roots.

When $x = +3$ and -3, $y = 4$ and -4;
$(3, 4)$, $(-3, -4)$ are roots.

When $x = 4$ and -4, $y = 3$ and -3;
$(4, 3)$, $(-4, -3)$ are roots.

When $x = 5$ and -5, $y = 2\frac{2}{5}$ and $-2\frac{2}{5}$;
$(5, 2\frac{2}{5})$, $(-5, -2\frac{2}{5})$ are roots.

When $x = 6$ and -6, $y = 2$ and -2;
$(6, 2)$, $(-6, -2)$ are roots.

When $x = 7$ and -7, $y = 1\frac{5}{7}$ and $-1\frac{5}{7}$;

$(7, 1\frac{5}{12})$, $(-7, -1\frac{5}{12})$ are roots.

When $x = 8$ and -8, $y = 1\frac{1}{2}$ and $-1\frac{1}{2}$;

$(8, 1\frac{1}{2})$, $(-8, -1\frac{1}{2})$ are roots.

When $x = 9$ and -9, $y = 1\frac{1}{3}$ and $-1\frac{1}{3}$;

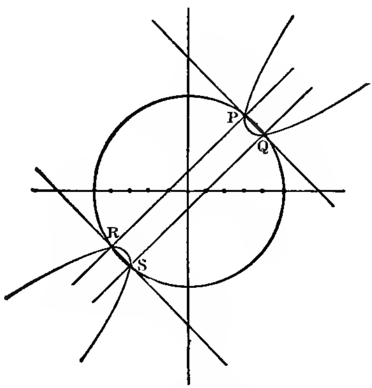
$(9, 1\frac{1}{3})$, $(-9, -1\frac{1}{3})$ are roots.

When $x = 12$ and -12, $y = 1$ and -1;

$(12, 1)$, $(-12, -1)$ are roots.

When $x = 24$ and -24, $y = \frac{1}{2}$ and $-\frac{1}{2}$;

$(24, \frac{1}{2})$, $(-24, -\frac{1}{2})$ are roots.



The graph of $xy = 12$ is the above hyperbola intersecting the graph of $x^2 + y^2 = 5$ in the points P, Q, R, and S, whose coördinates are the four roots of the equations.

In solving this example we get the equations $x + y = \pm 7$ and $x - y = \pm 1$. The graph of $x + y = 7$ is the line PQ, and that of $x + y = -7$ is the line RS. The graph of $x - y = 1$ is the line QS, and that of $x - y = -1$ is the line PR. These four lines intersect in the four points

P, Q, R, and S. The four lines have precisely the same intersections as the circle and hyperbola. This is why the set

$$\begin{cases} x+y=\pm 7, \\ x-y=\pm 1, \end{cases}$$

is equivalent to the set

$$\begin{cases} x^2+y^2=25, \\ xy=12, \end{cases}$$

as was used in the solution.

EXERCISES.

Solve the following sets of equations:

1. $\begin{cases} x^2+y^2=13, \\ xy=6. \end{cases}$

2. $\begin{cases} x^2+y^2=34, \\ xy=15. \end{cases}$

3. $\begin{cases} x+y=11, \\ xy=24. \end{cases}$

4. $\begin{cases} x+y=6, \\ xy=9. \end{cases}$

5. $\begin{cases} x-y=5, \\ xy=14. \end{cases}$

6. $\begin{cases} x+y=8, \\ x^2+y^2=34. \end{cases}$

7. $\begin{cases} x-y=1, \\ x^2+y^2=13. \end{cases}$

8. $\begin{cases} \dfrac{x}{4}+\dfrac{y}{5}=3, \\ \dfrac{x^2}{16}+\dfrac{9^2}{25}=5. \end{cases}$

9. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=7, \\ \dfrac{1}{x^2}+\dfrac{1}{y^2}=25. \end{cases}$

10. $\begin{cases} \dfrac{1}{x}-\dfrac{1}{y}=1, \\ \dfrac{1}{x^2}-\dfrac{1}{y^2}=7. \end{cases}$

11. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{6}, \\ \dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{13}{36}. \end{cases}$

12. $\begin{cases} 3x+5y=2xy, \\ xy=15. \end{cases}$

13. $\begin{cases} x^2+y^2=500, \\ x+y=30. \end{cases}$

14. $\begin{cases} x+y=a, \\ xy=\frac{1}{4}(a^2-b^2). \end{cases}$

Construct graphs for Exercises 4 and 7.

174. Special Methods; Higher Degrees. Simultaneous equations of higher degree than the second can frequently be solved by special methods. This is particularly true when they are symmetrical.

A few of the special methods will be illustrated. In such problems the student is expected to devise his own methods.

I. Solve $\begin{cases} (1)\ x^5 - y^5 = 211, \\ (2)\ \quad x - y = 1. \end{cases}$

$$(1) \div (2) = (3)\quad x^4 + x^3y + x^2y^2 + xy^3 + y^4 = 211.$$

$$(2)^4 = (4)\quad x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 = 1.$$

$$(3) - (4),\quad (5)\quad 5x^3y - 5x^2y^2 + 5xy^3 = 210.$$

$$(6)\quad x^3y - x^2y^2 + xy^3 = 42.$$

$$(7)\quad xy(x^2 + y^2) - x^2y^2 = 42.$$

From (2),
$$x^2 - 2xy + y^2 = 1.$$
$$x^2 + y^2 = 2xy + 1.$$

Substituting in (7),
$$xy(2xy + 1) - x^2y^2 = 42.$$
$$x^2y^2 + xy - 42 = 0.$$
$$(xy + 7)(xy - 6) = 0.$$
$$xy = 6 \text{ and } -7.$$
$$x = \frac{6}{y} \text{ and } \frac{-7}{y}.$$

Substituting in (2),

$$\frac{6}{y} - y = 1, \qquad \text{and} \qquad \frac{-7}{y} - y = 1,$$

$$6 - y^2 = y, \qquad\qquad -7 - y^2 = y,$$

$$y^2 + y - 6 = 0, \qquad\qquad y^2 + y + 7 = 0,$$

$$(y - 2)(y + 3) = 0, \qquad\qquad y = \frac{-1 \pm \sqrt{-27}}{2}.$$

$$y = 2 \text{ and } -3;$$

The corresponding values of x are 3, -2, $\frac{+1 \pm \sqrt{-27}}{2}$.

The roots are (3, 2), (-2, -3),

$$\left(\frac{1+\sqrt{-27}}{2}, \frac{-1+\sqrt{-27}}{2}\right), \left(\frac{1-\sqrt{-27}}{2}, \frac{-1-\sqrt{-27}}{2}\right).$$

II. Solve
$$\begin{cases} (1) & \dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{45}{4}, \\ (2) & \dfrac{1}{x}-\dfrac{1}{y}=\dfrac{3}{2}. \end{cases}$$

$$(2)^2=(3) \quad \frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=\frac{9}{4}.$$

$$(1)-(3)=(4) \quad \frac{2}{xy}=\frac{36}{4}.$$

$$(1)+(4)=(5) \quad \frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}=\frac{81}{4}.$$

$$(6) \quad \frac{1}{x}+\frac{1}{y}=\frac{\pm 9}{2}.$$

Combining (2) and (6) by addition,

$$\frac{1}{x}=3 \text{ and } -\frac{3}{2}.$$

$$x=\tfrac{1}{3} \text{ and } -\tfrac{2}{3}.$$

$$y=\tfrac{2}{3} \text{ and } -\tfrac{1}{3}.$$

III. Solve
$$\begin{cases} (1) & x^4+y^4=641, \\ (2) & x-y=7. \end{cases}$$

Put
$$x=u+v,$$
$$y=u-v.$$

$$x-y=2v=7.$$

$$v=\tfrac{7}{2}.$$

$$x^4+y^4=2u^4+12u^2v^2+2v^4=641.$$

Putting in the value of v,

$$2u^4 + 12u^2(\tfrac{49}{4}) + 2(\tfrac{49}{4})^2 = 641.$$

$$2u^4 + 147u^2 - \tfrac{2727}{8} = 0.$$

$$u^2 = \tfrac{9}{4} \text{ and } -\tfrac{303}{4}.$$

$$u = \pm\frac{3}{2} \text{ and } \pm\sqrt{\frac{-303}{4}}.$$

$$x = u + v = 5,\ 2,\ \frac{7}{2} \pm \sqrt{\frac{-303}{4}}.$$

$$y = u - v = -2,\ -5,\ -\frac{7}{2} \pm \sqrt{\frac{-303}{4}}.$$

IV. Solve $\begin{cases} (1) & x^2 + y^2 = 10, \\ (2) & xy - x - y = -1. \end{cases}$

Multiply (2) by 2 and add to (1),

(3) $x^2 + 2xy + y^2 - 2(x + y) = 8.$

(4) $(x + y)^2 - 2(x + y) - 8 = 0.$

(5) $(x + y - 4)(x + y + 2) = 0.$

(6) $x + y = 4$ or $-2.$

From (2) by substituting the value of $x + y$,

(7) $xy = 3$, or $-3.$

Multiply (7) by 2 and subtract from (1),

(8) $x^2 - 2xy + y^2 = 4$, or $16.$

(9) $x - y = \pm 2$, or $\pm 4.$

Combine (6) and (9),

$$x = 3,\ 1,\ 1,\ -3;$$

$$y = 1,\ 3,\ -3,\ 1.$$

The roots are $(3, 1)$, $(1, 3)$, $(1, -3)$, $(-3, 1)$.

EXERCISES.

Solve the following sets of equations:

1. $\begin{cases} \dfrac{3}{x} - \dfrac{1}{y} = 1, \\ 10\,xy = 1. \end{cases}$

2. $\begin{cases} x - \dfrac{y}{2} = 3, \\ xy - y^2 = 4. \end{cases}$

3. $\begin{cases} x + 7\,y = 15, \\ xy = 2. \end{cases}$

4. $\begin{cases} x^3 + y^3 = 9, \\ x + y = 3. \end{cases}$

5. $\begin{cases} x^3 - y^3 = 19, \\ x - y = 1. \end{cases}$

6. $\begin{cases} x^3 + y^3 = 91, \\ x + y = 7. \end{cases}$

7. $\begin{cases} x^3 + y^3 = 28, \\ x^2 - xy + y^2 = 7. \end{cases}$

8. $\begin{cases} x^2y + xy^2 = 30, \\ x + y = 5. \end{cases}$

9. $\begin{cases} (x + y)^2 + 4\,(x + y) = 45, \\ x - y = 1. \end{cases}$

10. $\begin{cases} (x - y)^2 + 3\,(x - y) = 18, \\ x + y = 7. \end{cases}$

11. $\begin{cases} x^2 + y^2 + x + y = 32, \\ xy = 12. \end{cases}$

12. $\begin{cases} x^4 + y^4 = 97, \\ x - y = 1. \end{cases}$

13. $\begin{cases} x^4 + y^4 = 82, \\ x + y = 4. \end{cases}$

14. $\begin{cases} x^2 + 2\,y^2 = 54, \\ xy + y^2 = 35. \end{cases}$

15. $\begin{cases} 3\,x^2 + 5\,y^2 = 17, \\ x^2 + 4\,xy + 3\,y^2 = 15. \end{cases}$

16. $\begin{cases} x^3 + y^3 = 28\,xy, \\ x + y = 12. \end{cases}$

17. $\begin{cases} x^4 + y^2 = 85, \\ x^2 + y = 11. \end{cases}$

18. $\begin{cases} x^2 + y^2 = 37, \\ x + y + xy = 13. \end{cases}$

19. $\begin{cases} x^3 - y^3 = 7\,xy, \\ x - y = 2. \end{cases}$

20. $\begin{cases} x^3 + y^3 = 35, \\ (x + y)\,(x^2 + y^2) = 65. \end{cases}$

EXERCISES.

1. Find two numbers whose difference is 5 and the difference of whose squares is 145.

2. The difference of two numbers multiplied by the greater $= 100$, but multiplied by the less $= 84$. Find the numbers.

3. The sum of two numbers is 7, and the sum of their cubes is 91. Find the numbers.

4. The product of the sum and difference of two numbers is 96, and the sum of their squares is 146. Find the numbers.

5. The sum of two numbers multiplied by their product is 120; and their difference multiplied by their product is 30. Find the numbers.

6. The difference of two numbers is 3, and the difference of their cubes is 117. Find the numbers.

7. The sum of the areas of two square fields is 2500 square rods; the sides of the fields are to each other as 3 to 4. Find the area of each field.

8. If the length and width of a rectangular field are each increased 10 rods, the area is increased 5 acres. But if the dimensions are each decreased 10 rods, the area will be $2\frac{1}{2}$ acres. Find the dimensions of the field.

9. The diagonal of a rectangle is 130 feet; the length of the rectangle is $2\frac{2}{5}$ times the width. Find the dimensions of the rectangle.

10. Find two numbers such that their product is 16 times their difference, and one of the numbers is double the other.

11. A rectangular lot containing 13200 square feet is surrounded by a walk 6 feet wide. The walk contains 3336 square feet. Find the dimensions of the lot.

12. In a certain number of two digits the sum of the squares of the digits is one more than twice their product, and the difference of the squares of the digits is 7. Find the number.

13. The fore wheel of a carriage makes 12 revolutions more than the hind wheel in going 240 yards; but if the circumference of each wheel is increased 1 yard, then the fore wheel will make only 8 revolutions more than the hind wheel in the same distance. Find the circumference of each wheel.

14. If a man had worked 5 days less and had received \$1 a day less, he would have earned \$30. If he had worked 10 days less, and had received \$2 a day more, he would have earned \$50. How many days did he work, and what were his wages a day?

15. If the numerator of a fraction be increased by 3 and the denominator be decreased by 3, the resulting fraction is the reciprocal of the first. If $\frac{17}{56}$ be added to the fraction, the sum is $\frac{1}{2}$ the reciprocal of the fraction. Find the fraction.

EXERCISES—MISCELLANEOUS.

1. Extract the square root of $x^2y^2 - axy^2 - (4x - \frac{1}{4}y)a^2y + 2a^3y + 4a^4$.

2. Find the roots by factoring:

(*a*) $x^2 - 7x = 30$.

(*b*) $x^2 + 7x = 60$.

(*c*) $y^2 - 9ay + 20a^2 = 0$.

(*d*) $(3y + 4)(2y - 3) - 39 = 0$.

(*e*) $y^2 - (c - a)(c - b) = (a - b)x$.

3. Determine whether 1, -1, $\frac{2}{3}$, or any one of them, is a root of $9x^2 - 3x = 2$.

4. Make an equation whose roots are $3 + \sqrt{7}$ and $3 - \sqrt{7}$.

5. Simplify $8\sqrt{3} + 13\sqrt{243} - 5\sqrt{121} + 4\sqrt{27}$.

6. Multiply $(x + \sqrt{x^{-1}})$ by $x^2 - \sqrt{x^{-2}}$.

7. Rationalize the denominator of $\dfrac{1 - \sqrt{5}}{2\sqrt{5} - 3\sqrt{6}}$.

8. Rationalize the denominator of $\dfrac{3 + \sqrt{-4}}{6 - \sqrt{-16}}$.

9. Divide $a - b$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.

10. Solve $\begin{cases} \dfrac{3}{x}+\dfrac{5}{y}=8, \\ \dfrac{1}{x}+\dfrac{2}{y}=4. \end{cases}$

11. Solve $dx^2 - x + c = 0$.

12. By means of the discriminant, tell what kinds of roots each of the following equations has:

(a) $3x^2 - 5x + 2 = 0$. (c) $5x^2 - 5x + 10 = 0$.

(b) $2x^2 + 11x - 10 = 0$. (d) $-x^2 + 4x + 2 = 0$.

13. Find two consecutive numbers whose product is 1260.

14. A number consisting of two digits which differ by 3, is 6 less than 7 times the sum of the digits. Find the number.

15. What value must a have to make the roots of $5x^2 - 11x + a = 0$ equal?

16. $\begin{cases} \dfrac{3}{x}-\dfrac{2}{y}=0, \\ \dfrac{9}{x^2}+\dfrac{4}{y^2}=\frac{1}{2}. \end{cases}$ Find x and y.

17. $\begin{cases} x^2 + x - y = 10, \\ 3y - x^2 + 3 = 0. \end{cases}$ Find x and y.

18. Two trains start at the same time to go 320 miles. One goes 8 miles an hour faster than the other and reaches its destination 2 hours sooner than the other. Find the rate of each train.

19. Solve $\dfrac{x+1}{x+2}+\dfrac{x+2}{x+3}+\dfrac{x+3}{x+5}=3$.

20. Express $x^{-5}y^{-\frac{4}{3}} + 2x^{\frac{3}{2}}y^{-\frac{3}{4}}$ without negative or fractional exponents.

21. $12x^4 - 17x^2 + 6 = 0$. Find x.

22. $y^3 - 3y^{\frac{3}{2}} = 88$. Find y.

23. A number consists of two digits. If its digits be inverted, the sum of the new and original number is 77 and their product is 1300. Find the number.

24. $36x^2 + 29ax + 5a^2 = 0$. Find x.

25. Make an equation whose roots are $\dfrac{a+\sqrt{2}}{3}$ and $\dfrac{a-\sqrt{2}}{3}$.

26. $\sqrt{x} + 2\sqrt[4]{x} = 8$. Find x.

27. $\dfrac{x+a}{b+a} - \dfrac{b-a}{x-a} = \dfrac{x+c}{b+c} - \dfrac{b-c}{x-c}$. Find x.

28. Find the roots of $(y-2)(y^2-12y+20)(y-1)=0$.

29. $5\sqrt{-x^2} \times 3\sqrt{-y^2} \times 2\sqrt{-9a^2} \times 3\sqrt{-4b^2} =$ what?

30. Multiply $\sqrt{3} - \sqrt{x} - \sqrt{y}$ by $\sqrt{3} + \sqrt{x} - \sqrt{y}$.

31. Perform the indicated operations and simplify the result:

$$\sqrt{\sqrt[3]{\left(\frac{a-b}{a+b}\right)^{12}\left(\frac{a+b}{a-b}\right)^{6}}}.$$

32. Square $a^{\frac{2}{3}} + b^{\frac{3}{4}} - c^{\frac{5}{2}}$.

33. Solve $11x - 11 = \dfrac{x^2+3}{2}$.

34. Solve $22 - 35x + 2x^2 = 0$.

35. Simplify $\dfrac{\dfrac{a+5}{a-5} + \dfrac{2a+3}{2a-3}}{\dfrac{a+5}{2a+3} - \dfrac{a-5}{2a-3}}$.

36. A path around the outside of a rectangular garden is 6 feet wide and 4224 square feet in area. The area of the garden is 28000 square feet. Find the dimensions of the garden.

37. Simplify $\dfrac{\dfrac{\frac{1}{y}+2}{\frac{1}{y}-2} - 1}{\dfrac{y+\frac{1}{2}}{y-\frac{1}{2}} - 1}$.

38. Solve $\dfrac{x^2+16}{25}+\dfrac{25}{x^2+16}=2.$

39. $\begin{cases} 3\,y^2-5\,xy+2\,x^2=14, \\ 2\,y^2-5\,xy+3\,x^2=\ \ 6. \end{cases}$ Find x and y.

40. Solve $\dfrac{y-6}{2}+\dfrac{2}{y-1}=\dfrac{y-1}{2}+\dfrac{2}{y-6}.$

41. Solve $(a^2-b^2)(x^2-1)=2\,x(a^2+b^2).$

42. Solve $\frac{1}{2}\left[x-\frac{1}{3}\left\{x-\frac{1}{4}\left(x-\dfrac{x-\frac{1}{6}\,x}{5}\right)\right\}\right]=53.$

43. What must be the value of x in order that $\dfrac{(x+3)^2}{3\,x^2+9\,x-5}$ may equal -1?

44. Find the value of $x^4-5\,x^3-12\,x^2-13\,x-7$, when $x=-\frac{1}{2}(1+\sqrt{-3})$.

45. Two rectangular fields each contain 10 acres. The perimeter of one is $\frac{1}{4}$ longer than that of the other. One of the fields is a square. What are the dimensions of each field?

46. If $ab+bc+ca=0$, prove that

(*a*) $(a+b+c)^2=a^2+b^2+c^2.$

(*b*) $(a+b+c)^3=a^3+b^3+c^3-3\,abc.$

(*c*) $(a+b+c)^4=a^4+b^4+c^4-4\,abc\,(a+b+c).$

CHAPTER XVII.

RATIO, VARIATION, AND PROPORTION.

I. RATIO.

175. *The* ***ratio*** *of a quantity A to a quantity B is the quotient of A by B.*

This quotient may be written in any one of the forms, $A \div B$, $\frac{A}{B}$, A/B, or $A:B$, each of which is read, the ratio A to B.

176. Ratio can exist only between two *abstract* numbers, or between two *concrete* numbers of the same kind. The ratio 5 to 7, or $\frac{5}{7}$, has a meaning, so does the ratio 6 bushels to 15 bushels, but not so with 6 bushels to 15 inches. Ratio merely expresses *the part one magnitude is of another.*

177. *The* ***terms*** *of a ratio are the numbers compared, the numerator being called the antecedent, the denominator the consequent.*

178. The ratio of antecedent to consequent is called a *direct ratio;* the ratio of consequent to antecedent is called an *inverse ratio.*

Thus, $14:28$ is direct, while $28:14$ is its inverse ratio. $\frac{b}{a}$ is the *inverse* of $\frac{a}{b}$.

179. A **compound ratio** is the product of two or more single ratios.

Thus, $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f}$ is the *compound ratio* of the single ratios, $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$.

180. Laws of Ratios. Since a ratio is a fraction, the operations which may be performed upon fractions may likewise be performed upon ratios. Below are enumerated the more important laws relating to ratios:

(1) A ratio is unchanged by multiplying or dividing both antecedent and consequent by the same number.

$$\frac{5}{8} = \frac{5 \times 4}{8 \times 4} = \frac{5 \div 2}{8 \div 2}; \quad \frac{A}{B} = \frac{mA}{mB} = \frac{A \div m}{B \div m}.$$

This law shows that the ratio of two concrete magnitudes of the same denomination is *independent of the unit of measurement.* The ratio of 2 miles to 5 miles is $\frac{2}{5}$; the ratio of 2 miles, expressed in feet, to 5 miles, expressed in feet, is $\frac{2 \times 5280}{5 \times 5280} = \frac{2}{5}$. The introduction of a multiplier in numerator and denominator of a ratio merely changes the *denomination* of the terms, if they be considered as concrete quantity, the ratio of the two magnitudes remaining unchanged.

(2) A ratio is changed by extracting the same root of each term of the ratio, or by raising each to the same power.

This law is true except in the case when antecedent and consequent are equal. If $\frac{A}{B} = m$, then will $\frac{\sqrt{A}}{\sqrt{B}} = \sqrt{m}$, and

$\frac{A^k}{B^k} = m^k$; but $\sqrt{m} \neq m$, and $m^k \neq m$. Hence, the ratio has been changed in all cases except that in which $m = 1$.

(3) A ratio is changed by performing *unlike* operations upon antecedent and consequent.

$\frac{5}{7} \neq \frac{5 \times 2}{7 \times 3}$, in which the antecedent has been multiplied by 2 and the consequent by 3. $\frac{A}{B} \neq \frac{mA}{nB}$, for $\frac{mA}{nB} = \left(\frac{m}{n}\right)\frac{A}{B}$.

(4) A ratio is changed by adding the same quantity to antecedent and consequent, except when the ratio is unity.

$$\frac{2}{5} \neq \frac{2+4}{5+4}; \quad \frac{A}{B} \neq \frac{A+x}{B+x}.$$

The truth of this law may be easily proven. Take the two fractions, $\frac{A}{B}$ and $\frac{A+x}{B+x}$, where x is any number whatever. By division,

$$\frac{A+x}{B+x} = \frac{A}{B} + \frac{x(B-A)}{B(B+x)},$$

which shows that $\frac{A}{B} \neq \frac{A+x}{B+x}$, except when $A = B$.

(5) A ratio is made more nearly equal to unity by adding any positive number to each of its terms.

Let $\frac{A}{B}$ be any ratio, and x any positive number. Then

$$\frac{A+x}{B+x} - 1 = \frac{A-B}{B+x},$$

and
$$\frac{A}{B} - 1 = \frac{A-B}{B}.$$

The first of these two differences is seen to be smaller than the second. Why? Hence, the truth of the law is established.

(1) Compare $\frac{2}{7}$ and $\frac{2+3}{7+3}$.

$$\frac{2}{7}=\frac{20}{70},\ \frac{2+3}{7+3}=\frac{5}{10}=\frac{35}{70}.$$

$\frac{35}{70}$ is more nearly 1 than $\frac{20}{70}$.

(2) Compare $\frac{5}{3}$ and $\frac{5+4}{3+4}$.

$$\frac{5}{3}=\frac{35}{21},\ \frac{5+4}{3+4}=\frac{9}{7}=\frac{27}{21}.$$

$\frac{27}{21}$ is more nearly 1 than $\frac{35}{21}$.

From these illustrations we may see that if a ratio be less than unity, the addition of the same positive number to the antecedent and consequent increases its value toward unity; and if the ratio be greater than unity, the addition of the same positive quantity to both antecedent and consequent diminishes the ratio toward unity.

181. The terms *ratio of less inequality*, *ratio of equality*, and *ratio of greater inequality* are sometimes employed to describe ratios less than unity, ratios equal to unity, and ratios greater than unity, respectively.

182. Limit. The result shown above,

$$\frac{A+x}{B+x}-1=\frac{A-B}{B+x},$$

indicates that the difference between $\frac{A+x}{B+x}$ and unity is a fraction $\frac{A-B}{B+x}$ whose value may be made as small as we please by making x sufficiently large. Hence, the

value of the ratio $\frac{A+x}{B+x}$, as x becomes infinitely large, approaches *unity*, which is called the *limit* of the ratio. The value that any algebraic expression continually approaches but never reaches is called its limit.

EXERCISES.

Write in their simplest forms the ratios of:

1. 625 to 125.
2. $480x$ to $120x^2$.
3. x^3+3x^2 to $x+3$.
4. $x^2-(y+z)^2$ to $x+y+z$.
5. x^3-y^3 to x^2+xy+y^2.
6. $\left(\frac{(a+b)^2}{4ab}-1\right)$ to $a-b$.

SUGGESTION. $\frac{x^3+3x^2}{x+3}=\frac{x^2(x+3)}{x+3}=x^2$.

7. $a^2-12a+20$ to $a-10$.
8. $6x^2+23ax+20a^2$ to $3x+4a$.
9. $x^4+x^2y^2+y^4$ to x^2-xy+y^2.
10. $(x^2+y^2)^2-4x^2y^2$ to $(x^2-y^2)^2$.

Write the compound ratios of the ratios:

11. 3 to 5 and 10 to 15.
12. $x+y$ to $x-y$ and x^2-y^2 to $(x+y)^2$.
13. 25 to x^3 and $3x^2$ to 50.
14. a^3-27b^3 to $(a-3b)^2$ and $a-3b$ to $a^2+3ab+9b^2$.
15. $(x+1)^2:(x^2+2x+1)$, $(x^3+1):(x+1)$,

and $(x-1):(x^2-x+1)$.

Find the value of x for which the ratio of:

16. 128 to x^2 is 2.
17. 625 to x^3 is 5.
18. $x+5$ to $x-1$ is 7.
19. $x^2+12x+5$ to x^2+5 is 3.
20. $(x+4):(3x+1)=\frac{1}{2}$.

Arrange the following ratios in descending order of magnitude:

21. $\frac{9}{20}, \frac{21}{32}, \frac{15}{26}, \frac{16}{27}, \frac{14}{25}, \frac{8}{19}, \frac{4}{15}, \frac{1}{12}$.

22. $\frac{a+5}{b+5}, \frac{a+1}{b+1}, \frac{a+3}{b+3}, \frac{a+7}{b+b}, \frac{a+4}{b+4}, \frac{a}{b}$.

II. VARIATION.

183. The term *variation* has little use in ordinary algebra, but its use is so frequent in physics that a brief treatment of the subject will be introduced here.

In physics we say "the weight of a uniform mass *varies* as the volume." This means that if W is the weight, and V the volume, then is $W = k \times V$, where k is a constant, the weight of a unit volume of any given substance.

In mensuration the circumference varies as the diameter. This means, that if C be the circumference and D the diameter of any circle, then will

$$C = k \times D,$$

k being a fixed constant for all circles. This constant is usually denoted by the Greek letter π; its numerical value is an incommensurable number, 3.14159 ....

184. In general a variable y is said to vary as another variable x, when

$$\frac{y}{x} = \text{a constant.}$$

The phrase, y varies as x, is sometimes written

$$y \propto x,$$

but is to be interpreted to mean

$$\frac{y}{x} = k, \text{ or } y = kx.$$

From this definition we see that a variation as here considered is equivalent to an equation.

185. Variations may be classified as follows:

(1) **Direct.** y varies *directly* as x, when

$$y = kx, \quad k = \text{a constant.}$$

The circumference of a circle varies directly as the radius.

$$C = kR, \qquad \text{where } k = 2\pi.$$

(2) **Inverse.** y varies *inversely* as x, when

$$y = \frac{k}{x}.$$

The volume of a gas varies inversely as the pressure, $V = \frac{k}{n}$, where V = volume and n = pressure.

(3) **Joint.** y varies *jointly* with x and z, when

$$y = kxz.$$

The weight of a rectangular parallelopiped of metal of unit height varies as the product of the length by the width,

$$W = k(l \times b).$$

k = weight of unit volume of the substance, l = length, and b = width of the rectangular solid.

(4) **Quadratic.** y varies as the square of x when

$$y = kx^2.$$

An example of such variation is found in the law of falling bodies; *i.e.*, the space fallen through by any body starting from rest equals a constant times the square of the time expressed in seconds.

$$S = kt^2, \quad k = \tfrac{1}{2}g, \quad g = 32 \text{ feet, } 2 \text{ inches.}$$

S = space described, t = time in seconds.

(5) **Direct and Inverse.** y varies directly as x and inversely as z when

$$y = k\frac{x}{z}.$$

An example of this form of variation is found in Newton's Law of gravitation. If M, m, be the masses of two attracting bodies, D their distance apart, and G the force of gravitation, then

$$G = k\frac{M \times m}{D^2}.$$

EXERCISES.

1. If $y \propto x$, and $y = b$ when $x = a$, find the value of y when $x = c$.

SOLUTION.

If $y \propto x$, then is $y = kx$. But $y = b$ when $x = a$;

hence, $$b = ka, \text{ or } k = \frac{b}{a}.$$

$$\therefore\ y = \frac{b}{a}x \text{ for any value of } x.$$

Hence, $$y = \frac{b}{a} \cdot c \text{ when } x = c.$$

2. If $y \propto x$, and if $y = 10$ when $x = 2$, find the value of y when $x = 12$.

3. If $y \propto x$, and if $x = 16$ when $y = 64$, find the value of x when $y = 15$.

4. The circumference of a circle varies as the radius ($C \propto R$). If $C = 3.1416$ when $R = \frac{1}{2}$, find the circumference of a circle whose radius is 12.

5. If $x \propto y$ and $w \propto z$, prove $\frac{x}{w} \propto \frac{y}{z}$.

6. If $x \propto y$ and $v \propto t$, prove $xv \propto yt$.

7. If $x \propto y$, prove that $x^n \propto y^n$.

8. If $x \propto y$ and $z \propto y$, prove that $(x^3 - z^3) \propto y^3$.

9. If y varies inversely as x^2, and if $y = 16$ when $x = 4$, find x when $y = 10$.

10. The volume of a sphere varies as the cube of its radius. If the volume of a sphere whose radius is 3 be 113.1, find the volume of a sphere whose radius is 20.

III. PROPORTION.

186. PROPORTION. *The equality of two ratios is called a proportion.*

Thus, $\frac{A}{B} = \frac{C}{D}$ is a proportion.

Various forms have been employed in writing a proportion, the following being the ones more frequently used:

$$A : B = C : D, \qquad A : B :: C : D,$$

$$A \div B = C \div D, \qquad \frac{A}{B} = \frac{C}{D}.$$

Each form is read A is to B as C is to D.

187. Proportionals. *The terms of the two ratios are called proportionals.*

In the proportion $A : B = C : D$, the terms A and D are called *extremes*, the terms B and C are called *means*, of the proportion.

In the proportion $A : B = B : D$, B is called the *mean proportional* to A and D; D is called a *third proportional* to A and B.

188. Theory of Proportion; Theorems. *A Theorem is a statement of a truth to be proved.*

A Corollary is a truth derived from the proof of a theorem.

The following theorems apply to proportions in which the terms of each ratio are considered abstract numbers.

THEOREM I. *In any proportion the product of the extremes equals the product of the means.*

Given $A : B = C : D$, or, more simply,

$$\frac{A}{B}=\frac{C}{D}.$$

Then is $AD = BC$. Why?

COROLLARY. *The mean proportional to two numbers equals the square root of their product.*

This corollary results from the above theorem by letting

$$C = B,\ \frac{A}{B}=\frac{B}{D},\ \text{or}\ AD = B^2,\ \text{whence}\ B=\sqrt{AD}.$$

THEOREM II. *If the product of two numbers equals the product of two other numbers, then either pair may be taken as extremes, and the other pair as means, of a proportion.* (*Inverse of Theorem* I.)

Given $AD=BC$.

Divide by BD, (1) $\frac{A}{B}=\frac{C}{D}$.

Divide by CD, (2) $\frac{A}{C}=\frac{B}{D}$.

Divide by AC, (3) $\frac{D}{C}=\frac{B}{A}$.

In each of the proportions (1), (2), (3), we have taken one pair of factors, A, D, or B, C, as extremes, the other as means.

THEOREM III. *If four numbers be in proportion, they are in proportion by inversion.*

Expressing this theorem algebraically,

$$\frac{A}{B}=\frac{C}{D}, \text{ from which } \frac{B}{A}=\frac{D}{C}.$$

Proof is left to the student. Result is easily shown true from Theorem I.

THEOREM IV. *If four numbers be in proportion, they will be in proportion by alternation; that is, the first is to the third as the second is to the fourth.*

Algebraically, if $A:B=C:D$, then is $A:C=B:D$. See (2) under Theorem II.

THEOREM V. *If four numbers are in proportion, they are in proportion when taken by composition; that is, the sum of the first and second is to the second as the sum of the third and fourth is to the fourth.*

This theorem stated algebraically is,

$$\text{if} \qquad \frac{A}{B}=\frac{C}{D}, \text{ then is } \frac{A+B}{B}=\frac{C+D}{D}.$$

Proof is easily derived by adding 1 to each member of the given proportion and reducing each member to a fractional form.

THEOREM VI. *In a series of equal ratios the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.*

Proof. Let the equal ratios be

$$\frac{A}{B}=\frac{C}{D}=\frac{E}{F}=\frac{G}{H}=\cdots=r,$$

where r is the common value of the ratios.

Then
$$\frac{A}{B}=r, \text{ or } A=r\times B,$$
$$\frac{C}{D}=r, \text{ or } C=r\times D,$$
$$\frac{E}{F}=r, \text{ or } E=r\times F,$$
$$\frac{G}{H}=r, \text{ or } G=r\times H,$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

Adding the equalities,
$$A+C+E+G+\cdots=r\times(B+D+F+H+\cdots).$$

Hence, $$\frac{A+C+E+G+\cdots}{B+D+F+H+\cdots}=r=\frac{A}{B}=\frac{C}{D}=\cdots.$$

If $$\frac{5}{15}=\frac{1}{3}=\frac{3}{9}=\frac{4}{12}=\frac{6}{18},$$
then is
$$\frac{1}{3}=\frac{5+1+3+4+6}{15+3+9+12+18}=\frac{3+4}{9+12}=\frac{5+1+6}{15+3+18},$$
and so on.

EXERCISES.

Find the value of the variable for which each of the following proportions is true:

1. $5:20=x:45.$
2. $x:37=x^2:26.$
3. $x:45=5:x.$
4. $(3x+4):(x+5)=(5x+1):(\frac{5}{3}x-4).$
5. $(4x-3):(2x+1)=(7x-4):(3x+2).$
6. Find a *fourth proportional* to 12, 16, and 40; also to a, b, and c.

7. Find a *mean proportional* to 16 and 49; also to l and m.

8. Find a *third proportional* to 25 and 35; also to x^2 and xz.

Are the following proportions true for all values of the letters:

9. $(9 - x^2) : (3 + x) = (3x - x^2) : x$?

10. $\left(\dfrac{x^2 + y^2}{2xy} - 1\right) : 2xy(x - y) = (x - y) : 4x^2y^2$?

11. $((x + y)^2 - z^2) : (x + y + z) = (x + y - z) : x$?

12. If $x : y = z : w$, show $\dfrac{lx + mz}{ly + mw} = \dfrac{x}{y}$.

SUGGESTION. Let $\dfrac{x}{y} = r$, $\dfrac{z}{w} = r$, then $x = ry$, $z = rw$, also $lx = lry$, $mz = mrw$. Add, $lx + mz = r(my + mz)$, etc.

13. If $\dfrac{x}{y} = \dfrac{z}{w}$, show $\dfrac{x^2 + z^2}{y^2 + w^2} = \dfrac{xz}{yw} = \dfrac{x^2}{y^2}$.

14. The rates of walking of two travelers are to each other as a to b. If one walk c miles in a given time, how far does the other walk in the same time?

15. The rear wheel of a wagon is a feet in circumference, the fore wheel is b feet in circumference. How often does the fore wheel rotate while the rear wheel makes m revolutions?

CHAPTER XVIII.

PERMUTATIONS AND COMBINATIONS.

I. PERMUTATIONS.

189. This subject can best be understood by introduction through a few concrete examples.

(1) How many different numbers of two digits each can be formed by using in every way any two of the five digits 5, 6, 7, 8, 9?

By writing any *one* digit first and each of the remaining *four* digits after it, we have the following *five* rows, each composed of *four* numbers:

56,	57,	58,	59,
65,	67,	68,	69,
75,	76,	78,	79,
85,	86,	87,	89,
95,	96,	97,	98.

In all there are $5 \times 4 = 20$ different numbers.

(2) How many different numbers of two digits each can be formed by using in every way any two of the four digits 5, 6, 7, 8?

Here we select any one of the four digits as the first, and place after it successively every one of the remaining three digits.

This gives the following numbers:

56,	57,	58,
65,	67,	68,
75,	76,	78,
85,	86,	87.

In all there are *four* selections of the first digit, and *four less one* selections of the second, giving $4 \times 3 = 12$ different numbers.

(3) How many different numbers of three digits each can be formed by using in every way any three of the five digits 5, 6, 7, 8, 9?

The first digit can be any *one* of the five; hence, the first place of each number can be filled in *five* different ways. *Four* digits remain to fill the other *two* places of each number. But we have just seen that *two* digits can be selected from *four* in $4 \times 3 = 12$ ways. Hence, with each of the *five* selections of the *first* digit, can be placed twelve selections of the digits filling the two remaining places. Hence, there are $5 \times 4 \times 3 = 60$ different numbers.

190. Definitions. (1) *The number of ways of selecting three things from a collection of five things is called the permutations of five things taken three at a time.*

(2) *The number of ways of selecting r objects from a collection of n distinct objects, regard being had for the order of selection, is called the permutations of n things taken r at a time.*

In this general case, n may be any number of objects, and r may be any integral number from 1 to n.

191. Symbol. Instead of writing the permutations of n things taken r at a time, the symbol ${}_nP_r$ is generally used.

Illustrations. (1) ${}_5P_2 = 5 \times 4$, the permutations of five things taken two at a time.

(2) ${}_4P_2 = 4 \times 3$, the permutations of four things taken two at a time.

(3) ${}_{10}P_3 = 10 \times 9 \times 8$, the permutations of ten things taken three at a time.

192. *Examples:* Let the pupil construct tables, if necessary, to verify the following results:

(1) ${}_3P_2 = 3 \times 2.$

(2) ${}_3P_1 = 3.$

(3) ${}_4P_3 = 4 \times 3 \times 2.$

(4) ${}_4P_4 = 4 \times 3 \times 2 \times 1.$

(5) ${}_5P_1 = 5.$

(6) ${}_5P_2 = 5 \times 4.$

(7) ${}_5P_3 = 5 \times 4 \times 3.$

(8) ${}_5P_4 = 5 \times 4 \times 3 \times 2.$

(9) ${}_5P_5 = 5 \times 4 \times 3 \times 2 \times 1.$

(10) ${}_{10}P_2 = 10 \times 9.$

The above examples indicate that there is a law governing the formation of permutations. It will be noted that the number of factors giving the permutations in each case equals the number of objects in each selection; the highest factor is the number to be permuted, and each succeeding factor is one less than the preceding. This

observation should lead one to some conclusion regarding the value of the general symbol

$$_nP_r.$$

We should expect the number of permutations of n things taken r at a time to be expressed by a product of r of the natural numbers beginning with n. Hence, we should find

(1) $_nP_1 = n.$

(2) $_nP_2 = n(n-1).$

(3) $_nP_3 = n(n-1)(n-2).$

(4) $_nP_4 = n(n-1)(n-2)(n-3).$

(5) $_nP_5 = n(n-1)(n-2)(n-3)(n-4).$

. . . .

(r) $_nP_r = n(n-1)(n-2)(n-3)\cdots(n-r+1).$

193. Value of $_nP_r$. To determine the number of permutations of n things taken r at a time, we may proceed as follows:

(1) Let the n distinct things be represented by n letters of the alphabet.

(2) Select any *one* letter to stand first in a set of words of two letters each. Then there would remain $n-1$ letters to fill the second place; but the first letter may be selected in n ways, and with each of these selections any one of the $n-1$ remaining letters may be placed.

(3) Hence, for the number of permutations of n things taken 2 at a time we have

$$_nP_2 = n(n-1).$$

(4) Let the first two letters of a three-lettered word be selected from the n letters; this selection can be made in

$n(n-1)$ ways, as shown in (3) above. Now we may select any one of the remaining $n-2$ letters to fill the third place.

(5) Hence, the formation of a three-lettered word from n letters can be accomplished in

$$_nP_3 = n(n-1)(n-2) \text{ ways.}$$

(6) In a similar manner we may show that

$$_nP_4 = {}_nP_3 \times (n-3) = n(n-1)(n-2)(n-3),$$
$$_nP_5 = {}_nP_4 \times (n-4) = n(n-1)(n-2)(n-3)(n-4),$$

and in general

$$\begin{aligned} _nP_r &= {}_nP_{r-1} \times (n-r+1) \\ &= n(n-1)(n-2)(n-3) \cdots (n-r+1). \end{aligned}$$

This general result may not be understood at the first reading of this subject, but its truth may be assumed until the pupil has had more experience.

Corollary I. When $r = n$, the general formula becomes

$$_nP_n = n(n-1)(n-2)(n-3) \cdots 4 \times 3 \times 2 \times 1,$$

a result easily remembered.

194. The Factorial Symbol. In the value of $_nP_n$ above, we have the product of the natural numbers from 1 to n. This product is often spoken of as "factorial n," and is written for brevity

$$\underline{|n}, \text{ or } n\,!.$$

Either $\underline{|n}$ or $n\,!$ is to be read factorial n, and means the product

$$n(n-1)(n-2)(n-3) \cdots 5 \times 4 \times 3 \times 2 \times 1.$$

EXERCISES.

Find values for:

1. $\underline{|5}$.

2. $\dfrac{\underline{|6}}{\underline{|5}}$.

3. $\dfrac{\underline{|10}}{\underline{|5} \times \underline{|4}}$.

4. $\dfrac{\underline{|20}}{\underline{|16} \times \underline{|2}}$.

5. $\dfrac{\underline{|16}}{\underline{|8}\,\underline{|8}}$.

6. $\dfrac{\underline{|16}}{\underline{|15}\,\underline{|4}}$.

7. $\dfrac{\underline{|20}}{\underline{|10}\,\underline{|12}}$.

8. $\dfrac{\underline{|25}}{\underline{|23}\,\underline{|2}}$.

9. $\dfrac{\underline{|40}}{\underline{|36}\,\underline{|4}}$.

10. $\dfrac{\underline{|8}\,\underline{|10}}{\underline{|15}}$.

195. Corollary II. When n objects are permuted all together, but are not all different, the number of distinct permutations is given by ${}_nP_n \div \underline{|s}$, where s is the number of objects which are alike.

Illustration. Required the number of different numbers obtainable from the five digits 5, 6, 6, 7, 8, taking five at a time.

If all digits be different, the number of selections would clearly be ${}_5P_5 = \underline{|5}$. But the two *sixes*, when permuted, give no new numbers; hence, all the permutations of the two *sixes*, *i.e.* $\underline{|2}$, must be excluded (divided out) from the total.

$$\therefore\ P = \frac{{}_5P_5}{\underline{|2}} = \frac{\underline{|5}}{\underline{|2}} = \frac{5 \times 4 \times 3 \times \cancel{\underline{|2}}}{\cancel{\underline{|2}}} = 60.$$

EXERCISES.

1. How many different numbers of three digits can be made from 1, 2, 3, 4, 5, 6?

2. How many different permutations can be made by taking 4 of the letters of the word *working*? By taking all of them?

3. Find the value of ${}_{16}P_3$; ${}_{17}P_4$; ${}_{20}P_5$.

4. How many permutations can be made from the 26 letters of the alphabet, taking 4 at a time?

5. How many six-place numbers can be formed from the Arabic numerals? (Include 0.)

6. In how many ways can a class of 6 be seated in a row of 6 chairs?

7. In how many ways can the front row of 6 chairs be filled from a class of 20?

8. How many different permutations can be made from the letters of the word *Indiana? Mississippi?*

9. How many even numbers of 6 places can be formed from the digits 1, 3, 4, 5, 7, and 9?

10. How many numbers between 50,000 and 60,000 can be formed from the digits 3, 4, 5, 6, 7?

11. In how many ways can 10 books be arranged on a shelf provided 2 particular books are always to be at the ends of the shelf?

12. In how many ways can 12 balls be arranged, if 5 are red, 4 white, and 3 blue?

II. COMBINATIONS.

196. (1) How many products of two factors each can be made from the five digits 5, 6, 7, 8, 9?

We have seen that the number of ways of selecting *two* things out of five is the permutations of five things taken two at a time. But in the case of products, $5 \times 6 = 6 \times 5$; hence, each arrangement of two digits is the result of a permutation of two things taken two at a time. These must all be excluded. Hence the number of products is

$$5 \times 4 \div \underline{|2} = 5 \times 4 \div 2 = 10.$$

(2) How many products of three factors each can be made from the five digits 5, 6, 7, 8, 9?

Since any three factors may be arranged in $3 \times 2 \times 1$ different ways, each arrangement giving the same product, we shall have to *divide out* $\underline{|3}$ of the permutations of five things taken three at a time. Hence, the total number of different products is

$$5 \times 4 \times 3 \div \underline{|3} = 20.$$

197. Definition. (1) *The number of ways of selecting three things from a group of five, no regard being had for the order of selection, is called the combinations of five things taken three at a time.*

(2) *In general, the number of ways of selecting r things from a group of n things, no regard being had for the order of selection, is called the combinations of n things taken r at a time.*

198. Symbol. Instead of the phrase, combination of n things taken r at a time, the symbol ${}_nC_r$ is usually employed.

Thus, ${}_5C_2$ is read, the combinations of five things taken two at a time; ${}_{10}C_4$ is read, the combinations of ten things taken four at a time, etc.

199. Relation between ${}_nC_r$ and ${}_nP_r$. It is easy to see that if we select from a given number of things any specified number, and do this in every possible way, having no regard to the order of selection, and then permute all the objects in each group in every way, we shall have the total permutations of the n things taken r at a time. The selec-

tions of the groups are the combinations, and the objects of each group are permuted r at a time; hence,

$$_nP_r = {_nC_r} \times {_rP_r}.$$

$$_nP_r = {_nC_r} \times \underline{|r}.$$

$$\therefore\ {_nC_r} = \frac{_nP_r}{\underline{|r}} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}}.$$

A second form for $_nC_r$ may be had by multiplying the numerator and denominator of the fraction on the right by $\underline{|n-r}$. This multiplier makes the numerator $\underline{|n}$, and the symbol $_nC_r$ becomes

$$_nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

$$_{12}C_7 = \frac{\underline{|12}}{\underline{|7}\,\underline{|12-7}} = \frac{\underline{|12}}{\underline{|7}\,\underline{|5}}.$$

From the first form

$$_{12}C_7 = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6}{\underline{|7}}$$

$$= \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot \underline{|5}}{\underline{|7}\,\underline{|5}}$$

$$= \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{\underline{|7}\,\underline{|5}}$$

$$= \frac{\underline{|12}}{\underline{|7}\,\underline{|5}}.$$

NOTE. When $r = n$, the second factor of the denominator $\underline{|n-r}$ becomes $\underline{|0}$, a symbol whose value is to be taken as *unity*. To show

$$\underline{|0} = 1,$$

we take the equality $\qquad \underline{|m} = m \times \underline{|m-1},$

and put $\qquad m = 1;$

then $\quad \lfloor 1 = 1 \times \lfloor 0,$

and as $\quad \lfloor 1 = 1, \ \lfloor 0$ must be 1.

$$\therefore \lfloor 0 = 1.$$

The form $\quad {}_nC_r = \dfrac{\lfloor n}{\lfloor r \ \lfloor n-r}$

is more easily remembered than the form

$${}_nC_r = \frac{n(n-1)(n-2)(n-3)\cdots(n-r+1)}{\lfloor r},$$

but the latter is especially useful in many applications.

EXERCISES.

1. How many combinations can be made from 9 things 3 at a time? 5 at a time?

2. Find the values of ${}_{10}C_4$, ${}_{12}C_9$, ${}_{12}C_3$.

3. In a meeting of 20 people, in how many ways can a committee of 5 be selected?

4. A school is composed of 19 boys and 25 girls. In how many ways can a committee consisting of 1 boy and 1 girl be selected?

5. From the above school, how many committees consisting of 2 boys and 1 girl can be selected?

6. From 15 persons, how many committees of 5 can be formed, provided *one* particular person is to be a member of every committee?

7. If out of 9 candidates there are to be 5 officers elected, how many different tickets can be formed?

8. From 4 vowels and 8 consonants, in how many ways can 5 letters be chosen, provided exactly 2 of them are vowels? Provided at least 2 of them are vowels?

9. How many even numbers of 4 places can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8?

10. ${}_nC_6 = {}_nC_{10}$; find n.

CHAPTER XIX.

SERIES.

200. General Definitions. (1) *Any set of numbers is an array, or succession.*

(2) *A series is a succession of numbers arranged according to some law.*

Thus, 1, 2, 3, 4, 5, 6, ⋯, is a series, the law of formation being that any number is to be had from the preceding by adding 1.

We may also define a *series* as a succession of numbers, the knowledge of two or more successive ones being sufficient to determine all.

Thus, 5, 7, 9, 11, ⋯, form a series, since by inspection of any two we see their difference to be 2; hence, any number of the series may be had from the preceding by adding 2.

(3) *The numbers forming a series are called the terms of the series.*

EXERCISES.

What law of formation exists in each of the following?

1. 2, 4, 6, 8, 10, ⋯.
2. 5, 10, 20, 40, 80, ⋯.
3. $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, $\frac{7}{2}$, 4, ⋯.
4. 3, $\frac{3}{2}$, $\frac{3}{4}$, $\frac{3}{8}$, $\frac{3}{16}$, ⋯.

5. $-1, 1, 3, 5, 7, 9, \cdots$.

6. $a,\ a+d,\ a+2d,\ a+3d,\ \cdots,\ a+(n-1)d$.

7. $a,\ ar,\ ar^2,\ ar^3,\ \cdots,\ ar^{n-1}$.

8. $5,\ -15,\ 45,\ -135,\ 405,\ -1215,\ \cdots$.

9. $x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \frac{x^9}{\lfloor 9} - \cdots$.

10. $1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \cdots$.

(4) *When the number of terms of a series is finite, the series is called a finite series.*

Thus, 2, 5, 8, 11, 14, is a finite series.

(5) *When the number of terms is infinitely great, the series is called an infinite series.*

Thus, if a series be formed by making any term the half of the preceding term and this process be continued indefinitely, as,

$$1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \tfrac{1}{16},\ \tfrac{1}{32},\ \cdots,$$

we have an *infinite series.*

(6) *If the sum of n terms of an infinite series can be shown to approach some finite number as n is made to approach infinity, the series is called convergent. If this sum can not be shown to approach some finite quantity, the series is called divergent.*

In the discussion of the subject of series, we shall examine only three special forms, the *arithmetical series*, the *geometrical series*, and the *binomial series*.

I. ARITHMETICAL SERIES (ARITHMETICAL PROGRESSION).

201. Definition. *An arithmetical series is a series in which the difference of any two successive terms is a constant.*

Illustrations. (1) 5, 9, 13, 17, 21, ⋯, is an arithmetical series, since the difference of any two successive terms is 4.

(2) 3, 3.5, 4, 4.5, 5, 5.5, ⋯, is arithmetical, since the difference is .5.

(3) a, $a+d$, $a+2d$; $a+3d$, $a+4d$, $\cdots$, $a+(n-1)d$, is arithmetical, since the difference of any two successive terms is d. In this illustration a is taken as any algebraic number, commensurable or not, and d likewise as any algebraic number.

202. Notation. We shall denote by a the first term of any arithmetical series, by d the constant difference (*common difference*) between any two successive terms, by l the last term or nth term, and by S the sum of n terms of the series.

203. Fundamental Formulas.

$$(1)\quad l=a+(n-1)d.$$

$$(2)\quad S=\frac{a+l}{2}\times n$$

$$=\frac{2a+(n-1)d}{2}\times n.$$

In these two relations five letters are involved, any two of which may be unknown.

The first of the above formulas is easily seen to be true from the manner of formation of the general arithmetical series shown in illustration (3) above.

To derive formula (2), we write the series

$$S = a + (a + d) + (a + 2d) + \cdots + (l - d) + l.$$

Then reverse the series,

$$S = l + (l - d) + (l - 2d) + \cdots + (a + 2d) + (a + d) + a,$$

and add the two equalities, giving

$$2S = (a + l) + (a + l) + (a + l) + (a + l) + \cdots$$
$$+ (a + l) + (a + l)$$
$$= n(a + l),\text{ since there are } n \text{ terms in the series.}$$

$$\therefore\ S = \frac{a + l}{2} \times n.$$

The second form of formula (2) is derived by replacing l by its value from formula (1).

204. Arithmetical Mean. If a, b, c be three successive terms forming an arithmetical series, b is called the *arithmetical mean* of a and c.

$b = \frac{1}{2}(a + c)$, for by the definition of the arithmetical series,

$$b - a = c - b;$$

(transposing), $2b = a + c$,

or $$b = \tfrac{1}{2}(a + c).$$

205. Arithmetical Means. In an arithmetical series a, b, c, d, e, f, $\cdots$, l, the terms b, c, d, e, f, $\cdots$, are called the *arithmetical means* of a and l.

206. *To insert k arithmetical means between any two numbers.*

Let a and b be any two numbers. After k means have been inserted, the whole series will consist of $k + 2$ terms.

Hence, b is the last or $(k+2)^{th}$ term of an arithmetical series of which a is the first, and d an unknown common difference.

Hence,
$$b = a + (k + 2 - 1)d$$
$$= a + (k+1)d,$$
$$d = \frac{b-a}{k+1}.$$

The common difference d being known, the series may be easily written thus:

$$a,\ a + \frac{b-a}{k+1},\ a + \frac{2(b-a)}{k+1},\ \cdots,\ b.$$

207. *An arithmetical series is determined when two of its terms are known.*

Let a be the kth and b be the mth term of an arithmetical series. Let x be the first term and y the common difference.

Then
$$a = x + (k-1)y,$$
$$b = x + (m-1)y.$$

By subtraction
$$b - a = (m-k)y,$$

or
$$y = \frac{b-a}{m-k} \text{ (the common difference),}$$

and $x = a - (k-1)\left\{\dfrac{b-a}{m-k}\right\} = \dfrac{(m-1)a - (k-1)b}{m-k}.$

The first term x and the common difference y being known in terms of a, b, k, m, the series may be written down.

Determine the arithmetical series in which the 5th term is 17, and the 12th term is 38.

SOLUTION.

Let $x =$ first term, and let $y =$ common difference.

Then $\quad l = a + (n-1)\,d$ becomes respectively,

$$\begin{cases} 17 = x + (5-1)\,y, \\ 38 = x + (12-1)\,y. \end{cases}$$

By subtraction, $\quad 21 = 7\,y$, or $y = 3$;

when $\quad y = 3,\ x = 5.$

Then the series is 5, 8, 11, 14, 17, 20, ⋯, 35, 38.

EXERCISES.

1. Find the 18th term of 2, 5, 7, 10, etc.

2. Sum 4, 7, 10, etc., to 9 terms.

3. Insert 5 arithmetical means between 10 and 34.

4. Find the 15th term of an arithmetical series whose 2d and 7th terms are 9 and 21, respectively.

5. Which term of the series 1, 6, 11, 16 is 96?

6. Find the sum of the natural numbers from 91 to 187.

7. Show that if any four numbers are in arithmetical progression, the sum of the 1st and 4th is the same as the sum of the 2d and 3d.

8. Find the 18th term of 27, 21, 15, 9, etc.

9. Find the sum of 12 terms of 3, $4\frac{1}{2}$, 6, $7\frac{1}{2}$, etc.

10. How many terms of 1, 2, 3, 4, etc., will make 465?

11. How many terms of 7, 11, 15, 19, etc., will make 297?

12. How many strokes does a clock strike in 12 hours?

13. Find the sum of all the even numbers from 100 to 200 inclusive.

14. Find the sum of all the numbers from 48 to 135 inclusive which are divisible by 3.

15. What debt could be paid in a year by the payment of 10 ¢ the 1st week, 40 ¢ the 2d week, 70 ¢ the 3d week, etc.?

16. Determine the series whose 10th term is 51 and whose 20th term is 101.

17. Determine a series whose 15th term is 0 and whose 31st term is 64.

18. Find the sum of all numbers from 105 to 361 inclusive, which, when divided by 4, leave a remainder of 1.

II. GEOMETRICAL SERIES (Geometrical Progression).

208. Definitions. *A geometrical series is a series in which the ratio of any two successive terms is a constant.*

Illustrations. (1) 2, 4, 8, 16, 32, is a geometrical series in which the ratio $16 \div 8 = 8 \div 4 = 32 \div 16 = 2$ is a constant.

(2) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \cdots$, is a geometrical series with ratio equal to $\frac{1}{3}$.

(3) $1, x, x^2, x^3, x^4, \cdots$, is a geometrical series with ratio equal to x.

(4) $a, ar, ar^2, ar^3, ar^4, \cdots, ar^{n-1}$, is a geometrical series in which the ratio is r.

209. Notation. The illustration (4) above suggests a notation for the geometrical series.

(1) $a =$ first term.

(2) $r =$ constant ratio.

(3) $l =$ last term, or nth term.

(4) $S =$ sum of n terms.

210. Formulas.

(1) $l = ar^{n-1}$.

(2) $S = a\left(\dfrac{r^n - 1}{r - 1}\right) = a\left(\dfrac{1 - r^n}{1 - r}\right)$.

(3) $S = \dfrac{a}{1 - r}$, when $r < 1$, and $n = \infty$.

The first of these formulas results from the law of formation of the series as indicated in illustration (4) above. The second formula we may derive as follows:

(1) $S = a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1}.$

Multiply by r,

(2) $Sr = ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} + ar^n.$

Subtracting (2) from (1),

(3) $S - Sr = a - ar^n.$

(4) $S(1 - r) = a(1 - r^n).$

(5) $$S = a \times \left(\frac{1 - r^n}{1 - r}\right).$$

Another method of derivation is worthy of attention. By actual division we know that

$$\frac{1 - r^3}{1 - r} = 1 + r + r^2,$$

$$\frac{1 - r^4}{1 - r} = 1 + r + r^2 + r^3,$$

$$\frac{1 - r^5}{1 - r} = 1 + r + r^2 + r^3 + r^4,$$

and so on; for the general case,

$$\frac{1 - r^n}{1 - r} = 1 + r + r^2 + r^3 + r^4 + r^5 + \cdots + r^{n-1}.$$

Now by writing the value of S again,

$$S = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-2} + ar^{n-1},$$

and factoring out a from each term on the right, we have

$$S = a(1 + r + r^2 + r^3 + \cdots + r^{n-2} + r^{n-1}).$$

The value in this bracket is the same as the value of $\frac{1 - r^n}{1 - r}$ above; hence,

$$S = a \times \left(\frac{1 - r^n}{1 - r}\right).$$

211. Sum to Infinity when $r < 1$. If the ratio r be less than unity, $r^n < 1$, and when $n \doteq \infty$, $r^n \doteq 0$. Hence, $s = \dfrac{a}{1-r}$, when $r < 1$, and $n \doteq \infty$.

The sign $\doteq$ is read approaches.

Illustration. Find the sum of $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ etc.

Here, $a = 1$, $r = \dfrac{1}{2}$, and $S = \dfrac{1}{1 - \frac{1}{2}} = 2$.

212. The Geometrical Mean. *If a, b, c, be three successive terms of a geometrical series, then b is equal to the square root of the product of a by c, and is the geometrical mean of a and c.*

By the definition of a geometrical series,

$$\frac{b}{a} = \frac{c}{b}; \text{ whence, } b^2 = ac, \text{ or } b = \sqrt{ac}.$$

213. Geometrical Means. *In a geometrical series the terms lying between any two terms are called the geometrical means of those two terms.*

Thus, 5, 10, 20, 40, 80, 160, are six terms of a geometrical series with ratio 2. The terms 10, 20, 40, 80, are the four geometrical means between 5 and 160.

214. Insertion of Geometrical Means. *Any number of geometrical means may be inserted between any two numbers.*

Proof. Let a and b be any two numbers, and let k be the number of means to be inserted. Then b is the $(k+2)^{th}$ term of a geometrical series, whose first term is a. If r be the unknown ratio,

$$b = ar^{(k+2-1)} = ar^{(k+1)}.$$

$$\therefore\ r = \left(\frac{b}{a}\right)^{\frac{1}{k+1}} = \sqrt[k+1]{\frac{b}{a}}.$$

Insert five geometrical means between 128 and 2.

These means may be written down if we know the value of the ratio. This is given by

$$r = \sqrt[k+1]{\frac{b}{a}}, \text{ where } k = 5,\ b = 128,\ a = 2.$$

$$\therefore\ r = \sqrt[6]{\frac{128}{2}} = \sqrt[6]{64} = 2.$$

Hence, the series is 2, 4, 8, 16, 32, 64, 128, and the means are 4, 8, 16, 32, 64.

215. A Geometrical Series Known. *A geometrical series is known when any two terms are known.*

Proof. Let a be the kth, and l the mth terms of a geometrical series; and let r be the unknown ratio, with x as first term.

Then $$a = xr^{k-1},$$
and $$l = xr^{m-1}.$$

These two equations are sufficient to determine the first term x and the ratio r.

By division $$\frac{xr^{m-1}}{xr^{k-1}} = \frac{l}{a},$$

or $$r^{m-k} = \frac{l}{a}.$$

(1) $$\therefore\ r = \sqrt[m-k]{\frac{l}{a}}.$$

Also, (2) $$x = \frac{a}{r^{k-1}} = \frac{a}{\left(\sqrt[m-k]{\frac{l}{a}}\right)^{k-1}} = \sqrt[m-k]{\frac{a^{m-1}}{l^{k-1}}}.$$

Equations (2) and (1) give the first term and ratio, respectively, of the required series.

EXERCISES.

1. Find the 10th term of 1, 2, 4, 8, etc.

2. Find the sum of the 10 terms in (1) above.

3. The 3d and 6th terms of a geometrical series are 27 and 729. Find the 8th term and the sum of the 8 terms.

4. Find the sum of 10 terms of 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc.

5. Sum 3, -3^2, 3^3, -3^4 to 8 terms.

6. A house with 8 windows was sold for \$1 for the 1st window, \$2 for the 2d, \$4 for the 3d, etc. What was received for the house?

7. If you receive \$5 Jan. 1, \$10 Feb. 1, \$20 March 1, and so on for each month of the year, what is the total amount you will receive during the year?

8. Find the sum to infinity of $\frac{3}{10}$, $\frac{3}{100}$, $\frac{3}{1000}$, etc.

9. Find the sum to infinity of 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, etc.

10. The arithmetical mean of two numbers is 13 and their geometrical mean is 12. Find the numbers.

11. Show that the series of alternate terms of a geometrical series is also a geometrical series.

12. If every term of a geometrical series is divided by the same quantity, the quotients form a geometrical series.

13. The reciprocals of the terms of a geometrical series form a geometrical series.

14. The difference between the 1st and 4th of four numbers in geometrical progression is 208, and between the 2d and 3d is 48. Find the numbers.

15. The sum of 3 numbers in geometrical progression is 14 and the sum of their reciprocals is $\frac{7}{8}$. Find the numbers.

III. BINOMIAL SERIES.

216. Definition. The series defined by

$$(a+b)^n = a^n + a^{n-1}b + \frac{n(n-1)}{\lfloor 2} a^{n-2}b^2 + \cdots$$

$$+ \frac{n(n-1)(n-2)(n-3)\cdots(n-r+1)}{\lfloor r} a^{n-r}b^2 + \cdots$$

is called the *binomial series.*

NOTE. In higher algebra it is shown that this series gives a true value of $(a+b)^n$ for all values of a and b *provided* n is an *integer*. It also defines $(a+b)^n$ properly when n is a negative number, or a fraction, provided $\frac{b}{a}$ be a proper fraction. The general proof of these assertions will not be attempted in this development.

EXERCISES.

1. $(1+x)^{30}$. In this $n=30$, $a=1$, $b=x$.

$$(1+x)^{30} = 1 + 30\,x + \frac{30 \times 29}{2} x^{29} + \frac{30 \times 29 \times 28}{1 \cdot 2 \cdot 3} x^{28} + \text{etc.}$$

2. Write out the first 6 terms of $(1+x)^{18}$; $(1-y)^{21}$; $(x+y)^{25}$.

3. Find the coefficient of x^{15} in $(1+x)^{30}$. (Referring to the binomial series, we see that $r=15$.)

4. $$(1-x)^{-3} = 1 + (-3)(-x) + \frac{(-3)(-3-1)}{\lfloor 2}(-x)^2$$

$$+ \frac{(-3)(-3-1)(-3-2)}{\lfloor 3}(-x)^3$$

$$= 1 + 3\,x + 6\,x^2 + 10\,x^3 + \text{etc.}$$

5. Write out the first 5 terms of

(*a*) $(1-x)^{-1}$; (*b*) $(1+x)^{\frac{1}{2}}$; (*c*) $(1-x)^{-\frac{3}{2}}$; (*d*) $(1+x)^{-\frac{2}{3}}$.

EXERCISES—MISCELLANEOUS.

1. Expand $(x+2y)^5$; $(2x-y)^6$; $(2a-3b)^7$.

2. Find the sum of 8 terms of $1,\ 2x,\ 4x^2,\ \cdots$.

3. Find the sum of 30 terms of $7,\ 11,\ 15,\ \cdots$.

4. Find the 6th term of $(1-2x)^{-3}$.

5. How many arithmetical means must be inserted between 10 and 40 so that the sum of the series may be 275?

6. Divide 26 into three parts which are in geometrical progression, and such that when 4 is added to the second part, the three parts are in arithmetical progression.

7. Insert a geometrical mean between 5 and 45. Explain the double sign of the result.

8. Sum to infinity $\frac{3}{2},\ \frac{2}{3},\ \frac{8}{27},\ \cdots$.

9. Find the coefficient of x^5y^5 in $(2x-3y)^{10}$.

10. Find the coefficient of x^8 in $(1-2x)^{-\frac{1}{2}}$.

11. In how many ways can 9 persons be selected from a party of 21 people?

12. How many committees consisting of 4 men and 3 women can be formed from 12 men and 10 women?

13. Each member of a baseball nine, except pitcher and catcher, can play in any position. In how many ways can the team be arranged upon the field?

14. In how many ways may a baseball nine be selected from 16 candidates, if 2 are pitchers, 3 are catchers, and the remainder can play in any position?

15. Expand $(4-2)^{\frac{1}{2}}$ to five terms, and thus get an approximate value of $\sqrt{2}$.

16. Expand $(100-1)^{\frac{1}{2}}$ to five terms, and thus get an approximate value of $\sqrt{99}$.

17. Expand $\left(1+\sqrt{\frac{a}{b}}\right)^6$.

18. Find the sum of all the even numbers between 205 and 341.

19. Find the sum to infinity of $1+\frac{1}{\sqrt{2}}+\frac{1}{2}+\frac{1}{2\sqrt{2}}+\cdots$.

20. The sum of the first three terms of a geometrical series is 21, and the sum of their squares is 273. Find the series.

21. A debt is to be paid by 10 payments which form an arithmetical progression. The third payment is \$220, and the seventh is \$360. Find the last payment and the total debt.

22. How many consecutive odd numbers beginning with 11 must be taken to make a sum of 759?

23. Prove that the squares of the terms of a geometrical series also form a geometrical series.

24. $(\sqrt{a+1}-\sqrt{a-1})^4=$ what?

25. Expand $(1+x+x^2)^5$.

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